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Some aspects of the theory of harmonic gauges over Riemann surfaces.

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Ph. D. Thesis .

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Introduction.

I present here my Ph.D. Thesis.

It consists of three articles, and of a final chapter, mainly consisting of complements, and scattered considerations.

The main subject of this Thesis is the study of harmonic maps from compact Riemann surfaces into unitary groups, and various generalisations and related subjects. Harmonic maps are critical points of the energy functional. In the case we are considering, the associated Euler-Lagrange equations are particularly simple, because of the conformal invariance of the energy for maps from surfaces, which emphasizes the role of the complex structure, and of the simplicity of the target manifold. Another important feature is that the non-linear equations are representable as 0-curvature conditions for families (loops) of connections. This is the elliptic version of a phenomenon which is typical of a class of evolution equations, where it induces soliton behaviour, and complete integrability.In this elliptic situation, this representation (due to Zakharov et al.) allows us to substitute algebraic geometry to analysis, in the description of the solutions.

The first part of this Thesis is the article "On the energy spectrum of harmonic 2-spheres in unitary groups", which has appeared in "Topology" Vol.27, No.2, 1988., but it dates back to October 1985. It is about harmonic maps from the Riemann sphere in the unitary group U(N). A holomorphic map from S^2 (considered with its standard complex structure) into some complex Grassmannian manifold $G_k(\mathbb{C}^N)$, via the Cartan embedding $G_k(\mathbb{C}^N) \dashrightarrow U(N)$ (which maps a complex subspace $\underline{p}$ of $\mathbb{C}^N$ into $1-2p \in U(N)$, where p is the hermitian projection onto $\underline{p}$) is a special case of such a harmonic map. More generally, using the 0-curvature representation, Uhlenbeck had described every harmonic map $f:S^2 \longrightarrow U(N)$ as a finite product of (essentially holomorphic) maps into complex Grassmannians, the so called "unitons":

$$f = f_0(1-2p_1)........(1-2p_k). \qquad f_0 \in U(N)$$

Moreover, she had associated to any harmonic map $f:S^2 \longrightarrow U(N)$ a holomorphic map (called extended solution) $G(\lambda)$ from the Riemann sphere into the group of based loops $\Omega U(N)$; so that to each uniton corresponded a map into the 1-parameter subgroups of $U(N)$, and the factorization of the original harmonic map corresponded to factorising the extended solution into monomials in the loop variable λ.

$$G(\lambda) = G_0(\lambda)\ \exp(itp_1)..........\exp(itp_k)$$

with $G(-1) = f$, $\lambda = \exp(it)$, and $G_0(\lambda) \in \Omega U(N)$.

Having previously remarked that the topological degree of Uhlenbeck's extended solution is a constant multiple of the energy of the original map (cf. chapter 4 in this Thesis), so that the energy of harmonic maps $S^2 \longrightarrow U(N)$ is "a priori" quantized, here we prove that, given any harmonic map $S^2 \longrightarrow U(N)$, any time we produce a new harmonic map, factoring out (or adding, in Uhlenbeck's terminology) a uniton, the energy decreases by the degree of the added uniton. Moreover, it is always possible to add unitons so that the new harmonic map has strictly smaller energy. To prove this, we use the well-known Birkhoff-Grothendieck theorem on holomorphic vector bundles over S^2, to find an appropriate subbundle with positive 1st Chern class.

In this way,starting from any harmonic map $f:S^2 \longrightarrow U(N)$, we can reach the simplest possible solution, a 0-energy constant map, after a finite number of steps: and going backwards, we have found a new, algebro-geometric proof of Uhlenbeck's factorization theorem.

The 2nd and 3rd article in this Thesis are essentially attempts to generalise the method and results above. The simplest possible generalization imaginable, is to substitute other compact Riemann surfaces for the Riemann Sphere. Unfortunately, while the 0-curvature representation, being local, still holds, the factorisation into unitons does not hold anymore. The main reason is that there is no analogue of the Birkhoff-Grothendieck decomposition theorem for holomorphic vector bundles over Riemann surfaces of higher genus; and moreover the negativity of the 1st Chern class of the tangent bundle of the surface doesn't allow enough unitons to exist.

The second part of this Thesis is the article :" Harmonic gauges on Riemann surfaces and stable bundles", to appear in 'Annales Institute Henri Poincare': Analyse non lineaire".

Starting from the remark that a harmonic map from a manifold M into U(N) may be considered as an element of the 'gauge group' (in the physicists' terminology) of unitary automorphisms of the trivial $\mathbb{C}^N$-bundle over M, we try to generalise the method and the results of the 1st article to the case of 'harmonic' elements of the gauge group of a hermitian complex vector bundle V over a compact Riemann surface.The gauge group acts on the space of connections on V, and we consider the energy functional on the gauge group, induced, once we choose a base point in the affine space of connections $\mathcal{A}$, by the natural, conformally invariant L^2 distance in this space. Allowing the base connection to vary, we finish up with considering pairs of connections 'harmonic one with respect to the other'. Limiting ourselves to the case of connections with central curvature, we get the system of equations:

$$\begin{aligned} &*(F(A) + 1/2\,(\Phi,\Phi)\,) = -2i\pi\mu(V) \\ &d_A\,\Phi = 0 \quad d_A\,(*\Phi) = 0 \end{aligned} \qquad (*)$$

where A is a unitary connection on V, of curvature F(A); $\mu(V) = c_1(V)/rk(V)$ is the normalised 1st Chern class of V; and Φ lies in the vector space, on which the space of connections is modelled. These equations, (in the case $\mu=0$), are restrictions of Yang-Mills equation with signature (++--) on conformally flat 4-manifolds.

The results we get generalize and clarify those in the 1st article.

I) It is possible to use unitons to generate solutions of (*): unitons are $\bar\partial_{\lambda}$-holomorphic, Φ_z-invariant subbundles.

II) On S^2, each solution of (*) is a finite product of unitons: therefore the energy $2|\Phi|^2$ of a solution (A,Φ) is an integral multiple of $1/8\pi$.

(II) On general Riemann surfaces, it is possible to factor out unitons, obtaining solutions with smaller energy, as long as the pair (A,Φ) is not semistable, in the sense of Hitchin (cf. references at the end of the article).

(iii) Unitons are critical points of the energy Hessian, restricted to a special class of variation: energy-decreasing unitons generate energy decreasing variations. Therefore there is a connections between Morse stability, and stability in a algebro - geometric sense.

The third article in this Thesis: "Some remarks about geodesics in gauge groups and harmonic maps" will appear in "Journal of Geometry and Physics". It consists mainly of examples and remarks, rather than deep theorems.

As before, we choose a unitary connection A on a hermitian vector bundle V over a Riemannian manifold M; and this induces a Riemannian (pseudo)-metric on the associated gauge group $\mathcal{G}$: this metric, which is right-invariant, but not bi-invariant is the pull-back of the L^2 metric on the space of connections $\mathcal{A}$, via the map $\mathcal{G} \dashrightarrow \mathcal{A}$ $g \mapsto g.A$. It is positive definite iff the connection A is irreducible; otherwise it is only semipositive definite.

If $t \mapsto g_t$ is a path in $\mathcal{G}$ and $F = g'g^{-1}$, then g_t is a geodesic if and only if:

$$d/_{dt} (\Delta_A F) + [\Delta_A F, F] = 0$$

Here we prove existence and uniqueness of solutions for the associated Cauchy problem (in the case the connection A is irreducible), and we show the existence of conserved quantities.

We then give two example of geodesics in a gauge group, more connected with the material in the other chapters of this Thesis.

The first example, in §7, connects harmonic maps from M into Grassmannian manifolds (and its twisted analogue of "harmonic subbundles' of the given bundle V over M) with 'stationary' geodesics of the associated gauge group. Because the metric on $\mathfrak{G}$ is not bi-invariant, not any 1-parameter subgroup of $\mathfrak{G}$ is a geodesic. Those which are geodesics are ' stationary' solutions (i.e. both the 2 terms in the geodesic equation above are 0). In particular, to each subbundle $\underline{\mathbf{p}} \subseteq V$, we associate, as usual, the hermitian projector operator p, the element of the gauge group 1-2p, and the one-parameter subgroup of $\mathfrak{G}$, $t \mapsto \exp(itp)$. Then $1\text{-}2p \in \mathfrak{G}$ is harmonic (or, in other words, the subbundle $\underline{\mathbf{p}}$ is harmonic) if and only if the associated 1-parameter subgroup is a (stationary) geodesic of the gauge group.

Let us consider an analogy: if we endow SO(3) with a right- but not bi-invariant Riemannian metric, then the geodesics describe the motion of a rigid body with a fixed point in $\mathbb{R}^3$. The stationary geodesics are those 1-parameter subgroups of SO(3), which describe pure rotations around one of the three main axes of inertia of the body.

In §8 we show a second example of geodesics in gauge groups: Uhlenbeck's extended solutions, described above. Indeed, a map from a manifold M into the loop group $\Omega U(N)$ may be seen as a loop in the space of unitary automorphisms of the trivial complex $\mathbb{C}^N$-bundle over M; therefore the construction in the paper by Uhlenbeck, and its twisted generalisation in the fourth chapter in this Thesis, provide loops in the gauge groups of complex bundles over Riemann surfaces; here we prove that they are geodesics.

Finally, we wonder if the geodesic equations we describe are 'completely integrable' (in the Hamiltonian sense), as in many cases of geodesics in finite dimensional Lie Groups.

The final chapter of this Thesis is meant to be a complement to the articles; it mainly consists of considerations and various minor results on related subjects, which were not important or complete enough to justify independent publication.

For example, we quote the basic facts about the geometry of the loop group $\Omega U(N)$; we show (without proof) the generator of the 1st cohomology group of the gauge groups over Riemann surfaces; we show that Uhlenbeck's factorisation is energy decreasing; we try to decompose general holomorphic maps $S^2 \to \Omega U(N)$ (which are in 1-1 correspondence with instantons over $\mathbb{R}^4$, by work of Atiyah and Donaldson) into product of unitons, with only partial success; and we make some remarks on the possible existence of a moduli space of solutions of the system of equations (•), or, more particularly, of harmonic maps from a compact Riemann surface into the unitary group.

About this last problem, we don' t even know the possible dimension of such a space, if it exists; and even recent, and quite refined, work by Hitchin (on harmonic maps from the 2-torus into SU(2)) doesn't seem to help.

The reader will find repetitions, and some change of notations between the different parts of this Thesis. This is due to their essential independence, and to the fact that they were written in different times. We ask the reader to forgive us.

In particular, some material concerning the variational origin of the system of equations (•), and its 0-curvature representation is present in both the second and third parts. This may be used, together with some part of the last chapter, to complement the first article, which doesn't treat these subjects, because it is mainly concerned with the proof of the factorization theorem, and it refers for the rest to the article by Uhlenbeck.

Acknowledgements. Several acknowledgements are distributed at the end of the articles. In particular, I would like to thank here: Mrs Sharon Laurenti (I.C.T.P. Trieste): the 1st article was typed on a word processor I had borrowed from her; and my supervisor Jim Eells, for many useful advices, but especially for the complete freedom in which I've always been allowed to work.

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4.

ON THE ENERGY SPECTRUM OF HARMONIC 2-SPHERES IN UNITARY GROUPS.

To appear on *"Topology"*.

On the energy spectrum of harmonic 2-spheres in unitary groups.

by:

Giorgio Valli

Introduction .

In [6] and in [11] a recursive procedure was given to produce harmonic maps from the Riemann sphere into the unitary group. This procedure, (called "adding a uniton" by K.Uhlenbeck), consisted in the multiplication by maps, associated with holomorphic vector bundles over S^2, the "unitons".(cf. §'s 1 ,2 ,3)

Here we show how adding a uniton makes the energy decrease by 8π the 1st Chern class of the added uniton.(cf. § 4)

Using as main tool a Birkhoff-Grothendieck decomposition for holomorphic vector bundles over S^2, (already used in [7]), we get as easy corollaries, in § 5 :

(i) a slightly different version of a factorization theorem, due to K.Uhlenbeck, of harmonic maps $S^2 \longrightarrow U(N)$ into finite products of unitons,

(ii) the integrality, modulo 8π, of the energy of such maps.

The above results are proved valid in the case of maps with values in $SU(N)$ (for $N \geq 3$) and in the complex Grassmannian as well.

Acknowledgements. We wish to thank K.Uhlenbeck, for permitting us to use some of her still unpublished material; J.Eells, J.Rawnsley, F.E.Burstall, J.C.Wood, (conversations with them were fundamental for

working out the results); more particularly, James Eells, for his constant encouragement; and the University of Warwick, for the excellent mathematical atmosphere.

Note. We have preferred to be quite sketchy in the exposition of the non-original parts, in order not to make this paper too long, with respect to the few new ideas actually involved. In particular, we have avoided the use of loop group machinery; it actually provides the motivation to everything, but it's not necessary for a self-contained exposition.

The reader may find details in [8] and in [13] (cf. also [1], [11])

Further developments. We wish to draw the attention of the reader to more recent work in this area by J.Wolfson ([14]), on harmonic maps from Riemann surfaces into Grassmannians, and by F.Burstall & J.Rawnsley ([13]), generalizing our methods and results to a certain class of Lie groups.

§ 1 Harmonic maps

Let $S^2 = CP^1$ be the 2-sphere, equipped with the standard Riemannian metric.

Let $G = U(N)$ be the unitary group in N dimensions ($N>2$), with its bi-invariant Riemannian metric, induced, via left translation, by the inner product $(A,B) = \mathrm{Tr}\, AB^*$ on its Lie algebra $\mathfrak{g} = u(N)$ of skew-hermitian NxN matrices.

We will study the energy functional

$$E(f) = 1/2 \int_{S^2} |f^{-1} df|^2 \qquad (1)$$

on the space of smooth maps $f: S^2 \longrightarrow U(N)$.

The critical maps for the energy functional are called harmonic

For any given map f, we define $A = 1/_2 f^{-1}df$ as 1/2 the pull-back of the Maurer-Cartan form of U(N); then we have:

$$E(f) = 2\int_{S^2}(A,A)$$

where the inner product is taken on $g\otimes T^*(S^2)$, and f is harmonic if and only if the following version ofthe Euler-Lagrange equations for E is valid:

$$d^*A=0$$

From now on, we shall look at S^2 as $S^2= \mathbb{C} \cup \{\infty\}$, using standard flat complex coordinates on C, for all the computations.This will work well because of the conformal invariance of the energy integral **(1)**.

With these conventions, if $f: S^2 \longrightarrow U(N)$, and $A = 1/2\, f^{-1}df = A_z dz + A_{\bar z} d\bar z$, with $-A_{\bar z} = A_z^*$, then we have

$$E(f) = 2\int_{S^2}(A,A) = -8\int_{C}Tr(A_z A_{\bar z}) \qquad \textbf{(2)}$$

The Euler-Lagrange equations for f, together with the f-pull-back of the Maurer-Cartan equation on U(N), are equivalent to the equation:

$$\bar\partial A_z + [A_{\bar z}, A_z] = 0 \qquad \textbf{(3)}$$

or to its complex conjugate:

$$\partial A_{\bar z} + [A_z, A_{\bar z}] = 0 \qquad \textbf{(4)}$$

where $\partial = \partial/\partial z$ $\bar\partial = \partial/\partial \bar z$ denote derivation operators on C.

§ 2 <u>The Grassmannian.</u>

Let us consider, on U(N), the involution $s \longrightarrow s^{-1}$. Its fixed set:

$$GR(N) = \{\, s \in U(N) \mid s^2 = I \,\} = \{\, s \in U(N) \mid s = s^* \,\} \quad ,$$

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consisting of unitary hermitian matrices, will be identified with the Grassmannian Kähler manifold of complex linear subspaces of C^N, via:

$s^2 = I$ if and only if $s = (p - p^{\perp})$

where p is the hermitian projection onto a complex subspace $\underline{p} \leq C^N$, and $p^{\perp} = I - p$ is the hermitian projection onto $(\underline{p})^{\perp}$.

The connected components of GR(N) are indexed by $k = \text{rank}\ \underline{p}$, each component corresponding to the usual Grassmannian $G_k(C^N)$ of k-dimensional planes in C^N.

The involution $s \longrightarrow -s$ on GR(N) induces canonical isomorphisms $G_k(C^N) = G_{N-k}(C^N)$ via $p \longrightarrow p^{\perp}$.

There is a canonical 1-1 correspondence between maps $f: S^2 \longrightarrow GR(N)$, and vector subbundles $\underline{p} \leq S^2 \times C^N$, via:

$f: S^2 \longrightarrow GR(N)$ if and only if $f = (p - p^{\perp})$

where p is the hermitian projection onto $\underline{p} \leq S^2 \times C^N$; and we have:

$f(S^2) \leq G_{\text{rank}\ p}(C^N)$

We call $\underline{p}$ __the bundle associated with the map__ f.

With our conventions, it follows that $\underline{p}$ is then the bundle associated with $-f$.

Given $f = p - p^{\perp}: S^2 \longrightarrow GR(N)$, then f is holomorphic if and only if $p^{\perp}\bar{\partial}p = 0$. This condition is equivalent to the holomorphicity of the associated bundle $\underline{p}$; and to the antiholomorphicity of $-f$, and of $\underline{p}^{\perp}$.

Given a map $f: S^2 \longrightarrow GR(N)$, we define the __topological degree__ **d(f)** of f, to be the algebraic degree of the induced map in cohomology:

$$f^*: H^2(G_k(C^N), Z) \cong Z \longrightarrow H^2(S^2, Z) \cong Z$$

(if f takes values in $G_k(C^N)$).

Equivalently, if ω is the Kähler form on $G_k(C^N)$, normalized so as to be the positive generator of $H^2(G_k(C^N), Z)$, then we have:

$$d(f) = \int_{S^2} f^*(\omega)$$

The following result is well known (see, for example, [3]).

Proposition 1. Let $f: S^2 \longrightarrow GR(N)$.Then we have:

$$d(f) = -c_1(\underline{p})$$

where $c_1(\underline{p})$ is the 1st Chern class of the bundle $\underline{p} \subseteq S^2 \times C^N$, associated with f.

□

§3 Unitons.

Recursive procedures to generate new harmonic maps, from Riemann surfaces into U(N) (or GR(N)),from given ones, have appeared recently in literature (cf.[3], [4], [7], [8], [11], [13], [14]). Here we examine what K. Uhlenbeck called "addition of a uniton".

Let $f: S^2 \longrightarrow U(N)$ be a given harmonic map. $A = 1/2\, f^{-1}df = A_z dz + A_{\bar{z}} d\bar{z}$ (in local conformal coordinates). Let us consider a new map of the form $f^{\sim} = f\,(p-p^{\perp})$, with $p-p^{\perp}: S^2 \longrightarrow GR(N)$. Then we have the following (cf. [8]).

Proposition (Uhlenbeck).If the equations

$$p^{\perp}A_z p = 0 \qquad \textbf{(5A)}$$

$$p^{\perp}(\bar{\partial}p + A_{\bar{z}}p) = 0 \qquad \textbf{(5B)}$$

are satisfied, then $f^{\sim} = f\,(p-p^{\perp})$ is harmonic as well. □

The proof of this proposition can be easily obtained by direct computation, but its real meaning lies hidden in the "loop group" construction in [8].

Equations **(5)** have a simple geometrical interpretation. Indeed, **(5B)** means that the bundle $\underline{p} \subseteq S^2 \times C^N$, associated with the map $p-p^\perp$, is holomorphic with respect to the $\bar\partial + A_{\bar z}$ derivative: if this happens, we say then that $\underline{p}$ is <u>$\bar\partial+A_{\bar z}$ - holomorphic</u>.

Contracting $A_z dz$, which is an invariantly defined 1-form on S^2, with any smooth vector field, we get a smooth section of $S^2 \times \mathcal{F} \subseteq S^2 \times End(C^N)$. Equation **(5A)** means that the bundle $\underline{p}$ is invariant under the action of any section obtained in this way: (we say that $\underline{p}$ is <u>A_z-stable</u>)

Following [8], we call the above procedure: <u>addition of the uniton</u> $\underline{p}$ to the map f. In particular, if f is a constant map, adding a uniton consists in the multiplication by an holomorphic map into a Grassmannian,(and any such holomorphic map is harmonic, cf.[5]).

Unitons are the basic (holomorphic) objects, because the "adding a uniton" procedure generates indeed every harmonic map $S^2 \longrightarrow U(N)$ (cf.[8], or corollaries 6 and 6', in the following).

<u>Remarks</u>:

(i) Subtracting and adding unitons are essentially the same procedures. if $f^\sim = f(p-p^\perp)$ is obtained adding the uniton $\underline{p}$ to f, (so that f can be regarded as obtained from $f^\sim$ by subtraction of $\underline{p}$), then it is possible to add the uniton $\underline{p}^\perp$ to $f^\sim$, producing $-f$.

(ii) If f is a map into a Grassmannian GR(N), (i.e., with our notations, $f:S^2 \longrightarrow U(N)$, and $f^2=I$), and $f^\sim = f(p-p^\perp)$ is obtained adding the uniton $\underline{p}$ to f, then f has also image contained in the Grassmannian if and only if $[f,p]=0$

§4 The energy

Let $f\colon S^2 \longrightarrow U(N)$ be an harmonic map; let $f^{\sim} = f\,(p - p^{\perp})$ be obtained from f by addition of the uniton $\underline{p}$; let $\Delta E_p = E(f^{\sim}) - E(f)$ be the energy increment in the procedure; and let $c_1(\underline{p})$ be the 1st Chern class of the associated bundle $\underline{p} \subseteq S^2 \times \mathbb{C}^N$.

Theorem 2 $\Delta E_p = -8\pi\, c_1(\underline{p})$.

Note. An analogous of this formula holds for maps into general real compact Lie groups (cf.[13]).

Proof.

Because of the conformal invariance of the energy integral, we can work out the computations on a conformal coordinate patch $\equiv \mathbb{C}$, as usual.

If $A^{\sim} = 1/2\, f^{\sim -1} df^{\sim} = \tilde{A}_z dz + \tilde{A}_{\bar z} d\bar z$ then we have:

$$\tilde{A}_z = A_z - \partial p$$

$$\tilde{A}_{\bar z} = A_{\bar z} + \bar\partial p \qquad \textbf{(6)}$$

So:

$$1/8\, E(f^{\sim}) = -\int_C \mathrm{Tr}(A_{\bar z} A_z) + \int_C \mathrm{Tr}(\bar\partial p\, \partial p) + \int_C \mathrm{Tr}(\partial p\, A_{\bar z}) - \int_C \mathrm{Tr}(\bar\partial p\, A_z)$$

$$1/8\, \Delta E_p = \int_C |\partial p|^2 + \int_C \mathrm{Tr}(\partial p\, A_{\bar z}) - \int_C \mathrm{Tr}(\bar\partial p\, A_z)$$

Integrating by parts, and using equations **(3)** and **(4)**, we get:

$$1/8\, \Delta E_p = \int_C |\partial p|^2 + 2\int_C \mathrm{Tr}(p\,[A_z, A_{\bar z}]) \qquad \textbf{(7)}$$

Now, because of $p^2 = p$, we have:

$$|\partial p|^2 = |p^\perp \partial p|^2 + |p \partial p|^2 = |p^\perp \partial p|^2 + |\partial p\, p^\perp|^2 = |p^\perp \partial p|^2 + |(\partial p\, p^\perp)^*|^2 =$$
$$= |p^\perp \partial p|^2 + |p^\perp \bar\partial p|^2$$

Using unitons equations **(5)**, we get:

$$|p^\perp \bar\partial p|^2 = |p^\perp A_{\bar z}\, p|^2 = -Tr(p^\perp A_{\bar z} p\, pA_z p^\perp) = -Tr(p^\perp A_{\bar z} pA_z) =$$
$$= -Tr(A_{\bar z} pA_z) + Tr(pA_{\bar z} pA_z) = -Tr(A_{\bar z} pA_z) + Tr(pA_{\bar z} A_z) =$$
$$= - Tr(A_{\bar z} p\, pA_z) + Tr(pA_{\bar z}\, A_z p)$$

$$|p^\perp \bar\partial p|^2 = |pA_{\bar z}|^2 - |A_{\bar z} p|^2 \qquad \textbf{(8)}$$

But we have:

$$Tr(p\,[A_z, A_{\bar z}]) = Tr(pA_z A_{\bar z}) - Tr(pA_{\bar z} A_z) = Tr(pA_z\, A_{\bar z} p) - Tr(pA_{\bar z}\, A_z p) =$$
$$= - |pA_z|^2 + |A_z p|^2 = - |p^\perp \bar\partial p|^2 \qquad \textbf{(9)}$$

So, substituting **(8)** and **(9)** in **(7)**, we get:

$$1/8\, \Delta Ep = \int_C (|p^\perp \partial p|^2 - |p^\perp \bar\partial p|^2) \qquad \textbf{(10)}$$

which is our main formula.

We now want to use the following theorem (cf.[5]):

<u>Theorem</u> <u>(Lichnerowicz)</u>. Let $h: M \longrightarrow N$ be a smooth map between compact Kahler manifolds.Then the quantity

$$\int_M (\|\partial' h\|^2 - \|\partial'' h\|^2) = \int_M h^*(\omega) \qquad \textbf{(11)}$$

(where ω is the Kähler form on N, and $\partial' h$, $\partial'' h$, denote the holomorphic and antiholomorphic differentials of h) <u>is an homotopy invariant</u> of h.

□

The condition for a map $(p-p^\perp): S^2 \longrightarrow GR(N)$ to be holomorphic,(or antiholomorphic), is, with our notations, $p^\perp \bar\partial p = 0$ (or $p^\perp \partial p = 0$) ; and we have :

$$|d(p-p^\perp)|^2 = 4\,|dp|^2 = 16\,|\partial p|^2 = 16\,(|p^\perp \partial p|^2 + |p^\perp \bar\partial p|^2) =$$

$$= 8\,(|p^{\perp}\partial p\, dz|^2 + |p^{\perp}\bar{\partial} p\, d\bar{z}|^2)$$

So, the holomorphic and anti-holomorphic differentials for maps $(p-p^{\perp}): S^2 \to GR(N)$ are $\sqrt{8}\, p^{\perp}\partial p\, dz$, $\sqrt{8}\, p^{\perp}\bar{\partial} p\, d\bar{z}$, on the chosen chart.

Using now proposition 1, and **(11)**, we conclude:

$$\Delta E_p = - C(k)\, c_1(\underline{p})$$

where C(k) is a constant, depending only on k = rank $\underline{p}$, originated by some previous constants, together with the non- integrality of the Kähler form on $G_k(C^N)$, corresponding to our chosen metric.

It is easy to establish: $C(k) = 8\pi$ for each k, writing down explicit holomorphic subbundles $\underline{p} \subseteq S^2 \times C^N$, and computing both sides of **(10)**. (For example, set $f=1$, $N=2$, and let

$\underline{p} = \{(z,v) \mid v=a\,(z,1),\ a\in C\} \cup \{(\infty,v) \mid v=(a,0),\ a\in C\} \subseteq S^2 \times C^2$

be the tautological line bundle, of 1st Chern class = −1 ; then we have:

$$\int_C (|p^{\perp}\partial p|^2 - |p^{\perp}\bar{\partial} p|^2) = \int_C 1/(1+z\bar{z})^2 = \pi$$

Choose now direct sums of copies of $\underline{p}$ with products $S^2 \times C^m$).

□

§ 5 <u>Factorization theorems for harmonic maps.</u>

Our proof of Uhlenbeck's factorization theorem strongly relies on the following, well known , theorem (cf. [2], [6]).

<u>Proposition 4</u> Every holomorphic vector bundle E over S^2 decomposes as a direct sum of holomorphic line bundles:

$$E = \bigoplus_{i=1}^{N} L^{k_i} \qquad \text{where} \quad c_1(L^k) = k$$

Such a (Birkhoff-Grothendieck) decomposition is not unique; but the associated Harder - Narasimhan filtration is uniquely defined for any given E:

$$\{ H^n = \bigoplus_{n \le k_i} L^{k_i} \}_{n \in Z}$$

and it is stable under holomorphic endomorphisms of E. Indeed, each H^n is spanned pointwise by the family of meromorphic sections of E, of divisor order $\geq$ n. □

Proposition 5 Let $f: S^2 \longrightarrow U(N)$ be a non-constant harmonic map.

Then there exists a harmonic map $f^{\sim} = f\,(p - p^{\perp})$, obtained by addition of a uniton $\underline{p}$ to f, such that:

(i) $\Delta E_p = E(f^{\sim}) - E(f) \le -8\pi$;

(ii) if $f: S^2 \longrightarrow GR(N)$, then $f^{\sim}: S^2 \longrightarrow GR(N)$ as well.

Moreover, it is possible to choose $f^{\sim}$ in a canonical way, so as to minimise ΔE_p.

Proof.

(i) Let us consider $A = {}^1/_2\, f^{-1} df = A_z dz + A_{\bar z} d\bar z$. We must show that there exists a vector subbundle $\underline{p} \subseteq S^2 \times C^N$, A_z-stable, $\bar\partial + A_{\bar z}$-holomorphic, and with positive 1st Chern class.

As remarked in §3, A_z does not define an endomorphism of the bundle $S^2 \times C^N$; but for every subbundle $\underline{p} \le S^2 \times C^N$, $A_z(\underline{p})$ is a well defined subbundle of $S^2 \times C^N$, not generally of constant dimension. But, if the subbundle p is $\bar\partial + A_{\bar z}$-holomorphic, then, because of equation **(3)**, $A_z(\underline{p})$ is $\bar\partial + A_{\bar z}$-holomorphic, and with fibers of constant dimension, except at isolated points. It may be proved (cf. [3], [8], [10]) that, because of the holomorphicity condition expressed in **(3)**, $A_z(\underline{p})$ can actually be completed into a proper $\bar{\partial}+A_{\bar{z}}$-holomorphic subbundle of $S^2\times C^N$, with fibers of constant dimension.

If we take as $\underline{p}$ a line bundle, spanned, except that at isolated points, by a meromorphic section s, then, contracting $A_z dz$ with a meromorphic vector field $v \neq 0$, and applying the result to s, we get a spanning meromorphic section for $A_z(\underline{p})$: and its divisor order, if $A_z(s)\not\equiv 0$, is at least equal to the order of s, +2, because v has divisor order at least 2 ($c_1(T(S^2))=+2$). We conclude that, if $\underline{p}$ is a line bundle, and $A_z(\underline{p})\not\equiv 0$, then we have: $\quad c_1(A_z(\underline{p})) \geq c_1(\underline{p}) + 2.$

We now consider $(H^n)_{n\in Z}$, the Harder-Narasimhan filtration of $S^2\times C^N$, with respect to the complex structure induced by $\bar{\partial}+A_{\bar{z}}$ (cf.[3]). We have: $A_z(H^n)\subseteq H^{n+2}$; in particular, each term H^n is A_z-stable.

We can now choose $\underline{p} = H^1$ as uniton; it has maximal positive 1st Chern class.

(ii) If $f^2 = I$, then, differentiating, we get:

$$\{f,A_z\} = 0 = \{f,A_{\bar{z}}\}$$

(where $\{X,Y\} = XY+YX$ is the anticommutator).

It follows that $[f,\bar{\partial}+A_{\bar{z}}] = 0$; so that f defines a holomorphic, unitary endomorphism of $S^2\times C^N$; and it stabilizes the Harder-Narasimhan filtration, commuting with the associated projection operator p.

$\square$

Remark Proposition 5 also holds for maps $f: S^2 \longrightarrow SU(N)$, for $N\geq 3$, but the choice of the uniton $\underline{p}$ is no more canonical.

Sketch of a proof.

If $f: S^2 \longrightarrow SU(N)$, and $f^{\sim} = f\,(p-p^{\perp})$, then we have:

f^{-} takes values in SU(N) if and only if rank $p^{\perp}$= N-rank p is even.

So, it's sufficient to show the existence of a subbundle of $S^2 \times C^N$, A_z-stable, $\bar\partial + A_{\bar z}$- holomorphic, of positive 1st Chern class, and of even rank, if N is even, or of odd rank, if N is odd. Excluding the case N=2, where non trivial unitons are associated to line bundles, and have determinant -1, and taking into account the fact that $A_z(H^n) \subseteq H^{n+2}$, so that, in particular, the highest terms of the filtration lie in the kernel of A_z, it will be easy for the careful reader to construct by hands a bundle p with the desired properties, in a non canonical way. □

The following is now obvious, by an induction argument, and some previous remarks.

Corollary 6 (Uhlenbeck's factorization theorem , revisited : cf. [8]).

Let $f: S^2 \longrightarrow U(N)$ be an harmonic map. Then there exist canonical harmonic maps $f_0, f_1, \quad, f_k : S^2 \longrightarrow U(N)$, with f_0 constant map, and f_k=f, such that:

(i) $\forall j$, $1 \leq j \leq k$: $f_j = f_{j-1}(p_j - p_j^{\perp})$

is obtained by addition of a uniton p_j

(ii) $0 = E(f_0)/8\pi \leq E(f_1)/8\pi - 1 \leq \qquad \leq E(f)/8\pi - k$

(In particular, $k \leq E(f)/8\pi$)

The same statements hold for maps f's taking values in the complex Grassmannian GR(N); and in SU(N) ($3 \leq N$), (but in this last case the maps f_j's are not canonically chosen). □

We can reformulate part of this result.

Corollary 6' Let $f: S^2 \longrightarrow U(N)$ be an harmonic map.

Then there exists a canonical factorization:

$f = Q\,(p_1 - p_1^{\perp}) \cdots\cdots (p_k - p_k^{\perp}) \qquad\qquad k \leq E(f)/8\pi$

where $Q \in U(N)$, and each p_j is the hermitian projection onto a subbundle of $S^2 \times C^N$, holomorphic with respect to the complex structure, induced by the operator $\bar\partial + \bar\partial p_1 + \ldots + \bar\partial p_{j-1}$ □

Remark: Unfortunately, our factorization is canonical, but not unique, unlike the one found by K.Uhlenbeck; actually, we do not know if they coincide.

As an easy consequence of the factorization theorem, and of the energy formula given by theorem 2, we can now get a new result: for harmonic maps $S^2 \to U(N)$, the energy gives a natural "topological charge":

Corollary 7 The energy of harmonic maps $S^2 \to U(N)$ can take as values only integral multiples of 8π. □

Note.

This last statement may also be obtained as an immediate consequence of a "Lax pair" construction (Zakharov et al.,1978; cf. also Uhlenbeck 1985), associating to each harmonic map f from the Riemann sphere into a real, compact, simple Lie group G, an (holomorphic) map F into a "loop group" ΩG. The topological degree of such an F is a well defined integer, ($\pi_2(\Omega G) \cong \pi_3(G) \cong Z$), which can be easily computed in cohomology, because the integral generator of $H^2(\Omega G, R)$ is a well known 2-form on ΩG (cf.[1]); such a degree is, up to multiplication by a constant, the energy of the original map f.

Details of this computation, of unclear paternity, will soon appear, hopefully (cf. also [13]).

Remark: corollary 7 is definitely false for harmonic maps defined on a generic Riemann surface M^2. As a simple counterexample, let us consider any homomorphism $f: T^2 \to U(N)$ (where $T^2 = C/Z+iZ$ is a flat torus); then the associated 1-form $A=1/2\, f^{-1}df$ is constant (so f is harmonic); and the energy is an integral multiple of π^2 .

At the same time, there appears to be no general procedure for describing all harmonic maps from a generic Riemann surface into U(N), or into the complex Grassmannian GR(N) (cf. also [14]).

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Harmonic gauges on Riemann Surfaces and stable bundles.

Harmonic gauges on Riemann surfaces and stable bundles.

Giorgio Valli

§ 0 Introduction.

Let $P \dashrightarrow M^2$ be a principal U(N)-bundle over a compact Riemann surface M^2 .

We study the equations:

$$\begin{cases} F(A) + {}^1/_2[\Phi, \Phi] = -2\, i\, \pi\, * \mu(P) & (*) \\ d_A \Phi = 0 \quad d_A * \Phi = 0 \end{cases}$$

where: A is a unitary connection on $P \dashrightarrow M^2$, of curvature F(A); and Φ is a section of $T^*(M) \otimes \mathrm{ad}\, P$ ($\mathrm{ad}\, P \dashrightarrow M^2$ is the "adjoint bundle" associated to $P \dashrightarrow M^2$ via the adjoint representation of U(N)), and μ is the normalised 1st Chern class of P.

Equations (*) generalise the harmonicity equations for maps $M^2 \dashrightarrow U(N)$, and they still maintain a variational origin (cf. § 3 and [V2]).

Here we generalise some previous work in this subject by Uhlenbeck [U] and Valli [V1], to this twisted situation: we show the existence of a recursive procedure (called "addition of a uniton" or "flag transform" in the literature cf. [U], [B-R]), which generates solutions of (*) by means of choices of appropriate holomorphic vector subbundles.

We prove that, on the Riemann sphere, this procedure generates all solutions of (*), starting from one with $\Phi=0$; while, on general surfaces M^2's, it fails to work, as long as we reach a pair (d_A, Φ), which is semistable in the sense of Hitchin (cf. [H 2]).

Our proof is based, as in [V 1], on progressive reduction of the energy $2\|\Phi\|^2$ of a solution (A, Φ); and on a topological expression for the decrease of energy in a flag transform.

Finally, we apply the same formula to show that the Morse (semi)-stability of a solution (A, Φ) of (*) implies the (semi)--stability of the holomorphic structure defined by the $\bar\partial$-operator $\bar\partial_A$; to do that, we split the energy Hessian, restricted to a special class of variations, into sum of two pieces, the difference of which is a topological term.

We regret that so much of the paper consists of preliminaries; our notation is anyway the "most standard" in current literature (in particular, cf.[A-B],[D],[H], and §1).

Finally, we wish to thank J.C.Wood and N.Hitchin for having informed us about their work.

§1 Dictionary

Let M^2 be a compact Riemann surface, and $P \longrightarrow M^2$ be a given smooth principal $U(N)$-bundle. Equivalently, we may consider the associated complex hermitian vector bundle $V \longrightarrow M^2$, associated to $P \longrightarrow M^2$ via the standard representation of $U(N)$ in $\mathbb{C}^N$ (we will generally prefer the vector bundle terminology throughout the following).

The adjoint bundle ad $P \longrightarrow M^2$ is the Lie algebra bundle associated to $P \longrightarrow M^2$ via the adjoint representation of $U(N)$ in $\mathfrak{u}(N)$; equivalently, it consists of the skew-hermitian elements of End (V).

Let $\mathcal{A} = \mathcal{A}(P)$ be the space of unitary connections on $P \longrightarrow M^2$.

Each connection $A \in \mathcal{A}(P)$ defines exterior differential operators d_A, defined on the space of sections of each of the bundles above.

Let $\mathfrak{G}$ be the "gauge group" of smooth automorphisms of the principal bundle $P \dashrightarrow M^2$. Its "Lie algebra" $\mathfrak{g}$ consists of smooth sections of the adjoint bundle ad $P \dashrightarrow M^2$.

The action of $\mathfrak{G}$ on P induces an action on the space of connections:

$$\mathfrak{G} \times \mathcal{A} \dashrightarrow \mathcal{A} \qquad (g,A) \longmapsto g_* A \tag{1}$$

defined by:

$$d_{g_* A} v = g^{-1} d_A (g v g^{-1}) g \tag{2}$$

for each v in $\mathfrak{g}$.

Finally, given a connection $A \in \mathcal{A}(P)$, we may split :

$$d_A = \partial_A + \bar{\partial}_A$$

using the complex structure on M^2.

§2 Stable bundles & stable pairs.

From a topological point of view, complex vector bundles $V \dashrightarrow M^2$

are classified by their rank, and by their 1st Chern class:

$c_1(V) \in H^2(M^2, \mathbb{Z}) \cong \mathbb{Z}$

(the last isomorphism being evaluation on the fundamental 2-cycle).

We define the normalized 1st Chern class of $V \dashrightarrow M^2$ to be

$$\mu(V) = c_1(V) / \mathrm{rk}\, V \qquad \textbf{(3)}$$

Moreover, if $\mathbf{p} \subseteq V$ is a complex vector subbundle, we then define, for later typographical convenience:

$$\sigma(\mathbf{p}) = \mathrm{rk}\, \mathbf{p}\, (\mu(V) - \mu(\mathbf{p})) \qquad \textbf{(4)}$$

Let now $V \dashrightarrow M^2$ be a given complex vector bundle.

By a theorem of Koszul & Malgrange (cf. [A-B]), each unitary connection A on V defines a unique holomorphic structure on V, such that, for any local section s, we have:

s is holomorphic $\Longleftrightarrow$ $\bar{\partial}_A s = 0$

More precisely, holomorphic structures on $V \dashrightarrow M^2$ are in 1-1 correspondance with $\bar{\partial}$ - operators $\bar{\partial}_A$ on it.

In the following, we shall indicate the complex vector bundle $V \to M^2$, equipped with the holomorphic structure induced by $\bar{\partial}_A$, with $(V, \bar{\partial}_A)$.

We recall now some standard definitions.

Def. 1. A holomorphic vector bundle $V \dashrightarrow M^2$ is called ***stable*** (resp. ***semistable***), if, for any proper holomorphic subbundle $\mathbf{p} \subset V$ we have: $\sigma(\mathbf{p}) > 0$ (resp. ≥ 0).

Def. 2 (cf. [H 2]). Let $V \dashrightarrow M^2$ be a holomorphic vector bundle, and let Φ_z be a holomorphic section of End (V) $\otimes$ K (where K is the canonical bundle of (1,0) forms). We say that (V, Φ_z) is a ***stable pair*** (resp. a ***semistable pair***) if for each proper holomorphic subbundle $\mathbf{p} \subset V$, which is Φ_z-invariant, we have $\sigma(\mathbf{p}) > 0$ (resp. ≥ 0).

We consider now the case $M^2 = \mathbf{C}P^1$. We then have a complete classification of holomorphic vector bundles.

Theorem (Birkhoff-Grothendieck).

(i) Each holomorphic vector bundle over $\mathbf{C}P^1$ splits as direct sum of holomorphic line bundles.

(ii) For each integer k, there exists one and only one holomorphic line bundle L^k over $\mathbf{C}P^1$ of 1st Chern class k (up to isomorphism).

Using this well known classification theorem, it' s easy to prove the following.

Lemma 1. Let V --> $\mathbf{C}P^1$ be a holomorphic vector bundle, and let Φ_z be a holomorphic section of End(V) $\otimes$ K . Suppose (V, Φ_z) is a semistable pair . Then:

(i) $\Phi_z = 0$;

(ii) $\mu(V)$ is an integer ;

(iii) V splits as direct sum of N= rk V copies of the line bundle $L^{\mu(V)}$.

Proof.

By the Birkhoff-Grothendieck theorem, we can split V into the direct sum of N line bundles. Let $\mathbf{p} \subseteq V$ be the holomorphic subbundle generated by the line bundles of highest 1st Chern class; then we have $\mu(\mathbf{p}) > \mu(V)$, unless $\mathbf{p} = V$.

By standard arguments involving the positivity of the 1st Chern class of the tangent bundle of S^2 (cf. [V 1]), $\mathbf{p}$ must lie in the kernel of Φ_z. In particular it is Φ_z -invariant, so that we must have, by the semistability assumption, $\mu(\mathbf{p}) \leq \mu(V)$; therefore $\mathbf{p} = V$. But this in turn implies $\Phi_z = 0$.

□

More generally, on Riemann surfaces of greater genus, stable pairs form a moduli space (cf. [H 2]).

§3 The equations

Let $V \dashrightarrow M^2$ be a complex hermitian vector bundle, $A \in \mathcal{A}$ a unitary connection on V, and Φ a skew-hermitian section of $End(V) \otimes T^*M^2$.

(Equivalently, $P \dashrightarrow M^2$ is the associated principal U(N)-bundle, $A \in \mathcal{A}(P)$, and Φ is a section of $ad\, P \otimes T^*M^2$).

We want to study the following system of equations:

$$\begin{cases} F(A) + 1/_2\, [\Phi, \Phi\,] = -2\pi i * \mu(V) \\ d_A \Phi = 0 \quad d_A{}^* \Phi = 0 \end{cases} \qquad (*)$$

Equivalently, if we decompose $\Phi = \Phi_z + \Phi_{\bar z}$, using the complex structure on M^2, then Φ_z is a section of $End(V) \otimes K$, and **(*)** is equivalent to **(**)**:

$$\begin{cases} F(A) + [\Phi_z\,, \Phi_{\bar z}\,] = -2\pi i * \mu(V) \\ \bar\partial_A\, \Phi_z = 0 \end{cases} \qquad (**)$$

We remark that equations **(*)** are <u>not</u> conformally invariant, because of the term $*\mu(V)$, which needs a volume form on M^2 in order to be defined. Alternatively, let $\mu^\sim$ be <u>any</u> 2- form on M^2, such that $\mu^\sim rkV$ is integral, and represents the 1st Chern class $c_1(V)$ (this constraint is due to Proposition 2 below, and to the Chern-Weil theory of characteristic classes). Then we can replace $*\mu(V)$ with $\mu^\sim$, in **(*)**, and throughout the following. And, up to a choice of $\mu^\sim$, equations **(*)** are conformally invariant.

Remark. Equations (*) are "gauge" invariant under the $\mathcal{G}$ - action:

$$A \longmapsto g_* A \qquad \Phi \longmapsto g_* \Phi = g^{-1}\Phi g \qquad \textbf{(5)}$$

The simplest possible example of solutions to (*) is the case when $\Phi=0$. Then (*) becomes:

$F(A) = -2\pi i * \mu(V)$

i.e. A is a connection with constant central curvature $-2\pi i * \mu(V)$. We can generalize this fact, as follows.

Let (A, Φ) be as above. We consider the loop of unitary connections on $V \dashrightarrow M^2$:

$$A_t = A + \cos t\ \Phi + \text{sen}\ t * \Phi \qquad \forall\, t \in [0, 2\pi] \qquad \textbf{(6)}$$

Equivalently:

$$\bar{\partial}_{A_t} = \bar{\partial}_A + \lambda^{-1}\ \text{ad}\ \Phi_z \qquad \lambda = e^{it} \in S^1 \qquad \textbf{(7)}$$

Proposition 2. The following statements are equivalent.

(i) (A, Φ) is a solution of (*).

(ii) A_t has constant central curvature $F(A_t) = -2\pi i * \mu(V)$ $\forall t \in [0,2\pi]$.

Proof.

$F(A_t) = F(A) + 1/2\, [\Phi, \Phi] + \cos t\ d_A \Phi + \text{sen}\ t\ d_A * \Phi$

□

We call a circle of unitary connections, with constant central curvature, of the form **(6)**, an "Uhlenbeck loop".

By proposition 2, Uhlenbeck loops are in natural bijective correspondance with the space of solutions of (*), modulo the S^1-action $\Phi_z \longmapsto e^{it}\, \Phi_z$.

The system of equations (*) has a simple variational origin. Suppose we take two unitary connections on $V \dashrightarrow M^2$; we call them

B, C (we should call them A_π, A_0, because of proposition 2, but we prefer to avoid proliferation of indexes).

We define B, C to be harmonic one with respect to the other (cf. [V 2]) if fixing one of them (say C) then the other (say B) is critical, with respect to the $\mathfrak{G}$ -action, with respect to the natural, conformally invariant, norm (Energy) on the space of connections $\mathcal{A}$:

$$E_C(B) = E_B(C) = 1/2 \, \| C - B \|^2 = 1/2 \int \mathrm{Tr}\,(C-B) \wedge *(C-B) \qquad \textbf{(8)}$$

$d/dt \, \| g_{t\,*} B - C \|^2 |_{t=0} = 0$ for each variation g_t :

$\forall g_t : (-\varepsilon, \varepsilon) \dashrightarrow \mathfrak{G} \qquad g_0 = 1$

Proposition 3. Let B, C be unitary connections with constant central curvature:

$$F(B) = F(C) = -2\pi i * \mu(V)$$

Then the followig statements are equivalent.

(i) B and C are harmonic one with respect to each other.

(ii) $A = 1/2\,(B + C)$ $\Phi = 1/2\,(B - C)$ are a solution of **(•)**.

Proof .

B and C are harmonic one with respect to each other if and only if $d_A{}^* \Phi = 0$.

And

$$F(B) = F(C) = -2\pi i * \mu(V) \quad \text{iff} \quad \begin{cases} F(A) + 1/2\,[\Phi,\Phi] = -2\pi i * \mu(V) \\ d_A \Phi = 0 \end{cases}$$

□

Remark. Let us consider the special case $\mathbf{V} = M^2 \times \mathbb{C}^N$; then the gauge group $\mathfrak{G}$ is $\mathfrak{G}$ = { smooth maps $M^2 \dashrightarrow U(N)$ }. Set C = 0, the zero connection; and, for $f \in \mathfrak{G}$ set $B = f_* C = f^{-1}df$; and A, Φ , as above. Then (A,Φ) is a solution of **(•)** if and only if f is an harmonic map $f: M^2 \dashrightarrow U(N)$ (cf. [H 1], [U], [V1]).

Proposition 4. Let (A,Φ) be a solution of **(*)**.

The second variation of energy is then:

$$H(u,u) = |d_A u|^2 - |[\Phi, u]|^2 \qquad \textbf{(9)}$$

Proof.

Let $B = A + \Phi$, $C = A - \Phi$, as above.

We consider a variation $g_t : (-\varepsilon, \varepsilon) \longrightarrow \mathfrak{G}$, $g_0 = I$; and let $B_t = g_{t*}B$. Then we have:

$$d/_{dt} |B_t - C|^2 = 2 < B_t - C, B - C >$$

$$d/_{dt^2} | B_t - C|^2 |_{t=0} =$$

$$= 2\{ | d_{Bt}(g^{-1}g')|^2 + < B_t - C, d_{Bt}(g^{-1}g')' + [d_{Bt}(g^{-1}g'), g^{-1}g'] > \}_{|t=0}$$

Set $u = g^{-1}g'_{|t=0}$. Then, because of the harmonicity equations, we have:

$$H(u,u) = 1/_2 \, d/_{dt^2} | B_t - C|^2_{|t=0} = |d_B u|^2 + < B-C, [d_B u, u] > =$$

$$= <d_B u, d_C u> = |d_A u|^2 - |[\Phi, u]|^2.$$

$\square$

§4 Adding unitons.

Let (A, Φ) be a solution of **(*)** on $V \dashrightarrow M^2$; and let A_t be the associated Uhlenbeck circle of connections **(6)**.

For each complex subundle $\mathbf{p} \subseteq V$, let p be the associated hermitian projection operator $p: V \dashrightarrow \mathbf{p}$, $p^2 = p$. Let us associate to $\mathbf{p}$ the closed 1-parameter subgroup of $\mathfrak{G}$:

$g_t = \exp(itp) \qquad t \in [0, 2\pi]$

Let $\quad A^{\sim}_t = g_{t*} A_t, \quad \forall t \in [0, 2\pi]$.

Proposition 5. The following statements are equivalent:

(i) $A^{\sim}_t$ is an Uhlenbeck loop;

(ii) $\underline{p}$ is a $\bar\partial_A$-holomorphic subbundle of V, and it is Φ_z- invariant.

Proof.

$A^{\sim}_t$ is certainly a loop of unitary connections with constant central curvature. We have to check when it is of the form **(6)**, for appropriate $\Phi^{\sim}$, $A^{\sim}$. Now, if $\lambda = e^{it}$, then we have:

$g_t = (p^{\perp} + \lambda p)$,

(where $p^{\perp} = I - p$ is the hermitian projection operator onto $(\underline{p})^{\perp} \subseteq V$); and :

$A_{\bar z,t} = A_{\bar z} + \lambda^{-1}\Phi_{\bar z}$; so that we have:

$$A^{\sim}_{\bar z,t} = (p^{\perp} + \lambda p)_* A_{\bar z} + \lambda^{-1}(p^{\perp} + \lambda^{-1}p)\Phi_{\bar z}(p^{\perp} + \lambda p)$$

And, using local coordinates it's easy to show that:

$A^{\sim}_{\bar z,t} =$

$$= \lambda(p^{\perp}\bar\partial_A p) + (p^{\perp}\bar\partial_A p^{\perp} + p\bar\partial_A p + p^{\perp}\Phi_{\bar z}p) + \lambda^{-1}(p\bar\partial_A p^{\perp} + p\Phi_{\bar z}p + p^{\perp}\Phi_{z}p^{\perp})$$
$$+ \lambda^{-2}(p\Phi_{\bar z}p^{\perp}) \qquad \mathbf{(10)}$$

But $A^{\sim}_t$ is of the form **(6)** if and only if $A^{\sim}_{\bar z,t}$ does not contain terms with λ, λ^{-2}.

$$\leftrightarrow \quad p^{\perp}\bar\partial_A p = 0$$
$$p\Phi_{\bar z}p^{\perp} = 0$$

$$\leftrightarrow \begin{cases} p^{\perp}\bar\partial_A p = 0 & \mathbf{(11\ a)} \\ p^{\perp}\Phi_z p = 0 & \mathbf{(11\ b)} \end{cases}$$

But **(11)** are the equations expressing (ii).

□

We call equations **(11)** "uniton equations" (cf. [U]).We call a subbundle $\underline{p} \subseteq V$, satisfying equations **(11)** (i.e. $\bar\partial_A$-holomorphic and Φ_z-invariant) a "uniton"; and we say that the new Uhlenbeck loop $A^{\sim}_t$ has been obtained from A_t by addition of the uniton $\underline{p}$ (cf.[U]), or by "flag transform" (cf.[B-R 1,2]).

Consistently with §3, given a solution (A,Φ) of **(•)**, we call $E = 2|\Phi|^2$ the ***energy*** of the solutions (A,Φ).

Proposition 5. Let (A, Φ) be a solution of **(•)** on $V \dashrightarrow M^2$; and let $(A^\sim,\Phi^\sim)$ be obtained from (A,Φ) by addition of the uniton $\mathbf{p} \subseteq V$. Then we have:

$$1/_2\, \Delta E = |\Phi^\sim|^2 - |\Phi|^2 = 2\pi\, \sigma(\mathbf{p}) \qquad \textbf{(12)}$$

Proof.

From **(10)** we get:

$$\Phi^\sim_z = -(p^\perp \partial_A p) + (p\Phi_z p) + (p^\perp \Phi_z p^\perp) \qquad \textbf{(13)}$$

Therefore:

$$|\Phi_z^\sim|^2 = |p^\perp \partial_A p|^2 + |p\Phi_z p|^2 + |p^\perp \Phi_z p^\perp|^2 \qquad \text{and:}$$

$$\begin{aligned} 1/2(|\Phi^\sim|^2 - |\Phi|^2) &= |\Phi_z^\sim|^2 - |\Phi_z|^2 = |p^\perp \partial_A p|^2 + |p\Phi_z p|^2 + |p^\perp \Phi_z p^\perp|^2 - |\Phi_z|^2 = \\ &= |p^\perp \partial_A p|^2 + |p\Phi_z p|^2 - |\Phi_z|^2 + |\Phi_z p^\perp|^2 - |p\Phi_z p^\perp|^2 = \\ &= |p^\perp \partial_A p|^2 - |p^\perp \Phi_z p|^2 - |p\Phi_z p^\perp|^2 = \\ &= (|p^\perp \partial_A p|^2 - |p\Phi_z p^\perp|^2) - (|p^\perp \bar\partial_A p|^2 - |p^\perp \Phi_z p|^2) \end{aligned}$$

(The last passage being possible because of the uniton equations **(11)**).

Proposition 5 follows then from lemma 6, in next §.

□

§5 A formula.

Let $V \dashrightarrow M^2$ be a complex hermitian vector bundle over the compact Riemann surface M^2, equipped with a hermitian metric; let A be a unitary connection on V, and $\Phi = \Phi_z + \Phi_{\bar z}$ a skew-hermitian section of $\mathrm{End}(V) \otimes T^*(M^2)$.

Lemma 6. Suppose (A,Φ) satisfy the 1st eq. of **(•)**:

$F(A) + 1/2\,[\Phi, \Phi\,] = -2\pi i * \mu(V)$

Let $\mathbf{p} \subseteq V$ be any complex subbundle of V, and let $p: V \longrightarrow \mathbf{p}$ be the associated hermitian projection operator. Then we have:

$$(\, |p^{\perp}\partial_A p|^2 - |p^{\perp}\Phi_z p\,|^2) - (\, |p^{\perp}\bar{\partial}_A p\,|^2 - |\,p^{\perp}\Phi_{\bar{z}} p|^2) =$$

$$= 2\pi\, \sigma(\mathbf{p}) \tag{14}$$

<u>Proof</u>.

Let us equipp $\mathbf{p}$ with the connection induced from A. If ∇ is the associated exterior differential, then we have, for each section v of p:

$\nabla v = p\, d_A v \;= d_A v - d_A p\; v$

And the curvature $F(\nabla)$ is:

$F(\nabla) = p\,(\, F(A) + d_A p \wedge d_A p\,)\, p$

By the Chern-Weil formulas for characteristic classes, we have:

$$-2i\pi\, c_1(\mathbf{p}) = \int \mathrm{Tr}\, F(\nabla) = \int \mathrm{Tr}(pF(A)\,) + \int \mathrm{Tr}(pd_A p \wedge d_A p) =$$

$$= -2i\pi \int \mathrm{Tr}(p) * \mu(V) \;- 1/2 \int \mathrm{Tr}\,[\Phi,\Phi\,]\, p + \int \mathrm{Tr}\,(pd_A p \wedge d_A p) \tag{15}$$

(using the hypotheses of the lemma).

But we have:

$$\int \mathrm{Tr}(p\; d_A p \wedge d_A p\,) = -i\,(\, |p^{\perp}\bar{\partial}_A p|^2 - |p\bar{\partial}_A p|^2\,) =$$

$$= \; -i\,(|p^{\perp}\bar{\partial}_A p|^2 - |p^{\perp}\partial_A p|^2) \tag{16}$$

(because $pd_A p = d_A p p^{\perp}$ and $p^{\perp} d_A p = d_A p p$); and:

$$1/2 \int \mathrm{Tr}\,[\Phi, \Phi]\, p \;= \int \mathrm{Tr}\,[\Phi_z, \Phi_{\bar{z}}]p \;= -i\,(\, |\Phi_z p|^2 - |p\Phi_z|^2) \;=$$

$$= \; -i\,(\, |p^{\perp}\Phi_z p|^2 - |p\Phi_z p^{\perp}|^2\,) \tag{17}$$

$$\int \mathrm{Tr}(p) * \mu(V) = \; rk(\mathbf{p})\; \mu(V) \tag{18}$$

Substituting **(16)**, **(17)** and **(18)** in **(15)**, we get:

$$-2i\pi\, c_1(\mathbf{p}) = \; (-2i\pi) rk(\mathbf{p})\; \mu(V) + \; i\,(|p^{\perp}\Phi_z p|^2 - |p\Phi_z p^{\perp}|^2) +$$

$$- \; i\,(|p^{\perp}\bar{\partial}_A p|^2 - |p^{\perp}\partial_A p|^2)$$

which is **(15)**. $\square$

§6 Main results.

Theorem 7 Let (A, Φ) be a solution of $(*)$ on $V \longrightarrow M^2$. Then there exists a solution of $(*)$ (A^0,Φ^0) such that :

(i) $\left((V,\bar{\partial}_{A^0}),\ \Phi_z^{\,0} \right)$ is a semistable pair;

(ii) (A,Φ) is obtained from (A^0,Φ^0) by a finite number of flag transforms, each one making the energy $E^1 = 2\,|\Phi^1|^2$ increase by a positive integral multiple of $8\pi/_{rk(V)}$.

Proof.

If $((V,\bar{\partial}_A),\Phi_z)$ is not a semistable pair, then there exists a $\bar{\partial}_A$-holomorphic subbundle $\mathbf{p} \subseteq V$, Φ_z-invariant, and with $\sigma(\mathbf{p}) < 0$.

Therefore $\mathbf{p}$ is a uniton, and we may add it to (A,Φ), to produce a new solution $(A^\sim,\Phi^\sim)$ of $(*)$, such that :

$\Delta E = 2\,(|\Phi^\sim|^2 - |\Phi|^2) = 4\,(|\Phi_z^\sim|^2 - |\Phi_z|^2) = 8\pi\ \sigma(\mathbf{p}) < 0$

(using proposition 5).

Moreover $\sigma(\mathbf{p}) = rk(\mathbf{p})\,(\mu(V)-\mu(\mathbf{p})\,) = rk(\mathbf{p})\,\mu(V) - c_1(\mathbf{p}) =$

$$= 1/_{rk(V)}\left\{ rk(\mathbf{p})c_1(V) - rk(V)c_1(\mathbf{p}) \right\}$$

Repeating this procedure, if necessary, we must eventually come to a stop, when we reach a semistable pair (A^0,Φ^0).

□

Theorem 8 Let (A, Φ) be a solution of $(*)$ on $V \longrightarrow \mathbb{CP}^1$. Then:

(i) $\mu(V)$ is an integer;

(ii) (A,Φ) is obtained from a solution (A^0,Φ^0), with:

$\Phi^0 = 0$

$F(A^0) = -2\pi i * \mu(V)$

by a finite number of flag transforms, each one making the energy $E^1 = 2|\Phi^1|^2$ increase by an integral multiple of $8\pi/rk(V)$.

(iii) V, with the $\bar{\partial}_{A^0}$-holomorphic structure, is a direct sum of N copies of the line bundle $L^{\mu(V)} \longrightarrow \mathbb{CP}^1$.

(iv) $E = 2\|\Phi\|^2 = 1/_{8\pi}\{a\,\mu(V) + b\}$ where $a \in \mathbb{N}$, $b \in \mathbb{Z}$.

Proof.

It easily follows from Theorem 7, lemma 1, and proposition 5.

□

Remarks. The flag transforms in theorems 7,8 may be chosen to be "canonical" in some sense. For example, we can choose at each step the most energy-decreasing uniton, which is the one generated by all the unitons $\mathbf{p}' \subseteq V$, with $\sigma(\mathbf{p}') > 0$. Another possible "canonical" choice, when $M^2 = \mathbb{CP}^1$, is to choose the image bundle (or the kernel bundle) of Φ_z at each step as uniton: it is not necessarily energy decreasing, but it arrives to the 0-energy solution, after a finite number of steps, as in theorem 8. For a more detailed analysis, cf. [V 3]. For a more explicit description of the factorization, when $M^2 = \mathbb{CP}^1$, with a unicity result, cf.[W].

Let (A,Φ) be a solution of **(•)** on $V \dashrightarrow M^2$. We want to use lemma 6 in order to study the energy hessian:

$$H(u,u) = \| d_A u\|^2 - \| [\Phi,u] \|^2 \qquad \textbf{(9)}$$

where $u \in \mathfrak{g}$ is a smooth skew-symmetric section of $\mathrm{End}(V) \dashrightarrow M^2$.

We consider infinitesimal variations of the form $u = ip$, where $p: V \dashrightarrow \mathbf{p}$ is the hermitian projection operator onto a complex subbundle $\mathbf{p} \subseteq V$; we may call this kind of variations Grassmannian variations .

Proposition 9. There exist two quadratic functionals $H^1(p,p)$, $H^2(p,p)$, on the space of Grassmannian variations, such that we have:

$$H^1(p,p) + H^2(p,p) = H(ip,ip)$$

$$H^1(p,p) - H^2(p,p) = 4\pi\,\sigma(\mathbf{p}) \qquad \forall \mathbf{p} \subseteq V \qquad \textbf{(19)}$$

Proof.

Define: $H^1(p,p) = 2(\|p^\perp \partial_A p\|^2 - \|p\Phi_z p^\perp\|^2)$

$H^2(p,p) = 2(\|p^\perp \bar{\partial}_A p\|^2 - \|p^\perp \Phi_z p\|^2)$

and apply lemma 6 □

Corollary 10 Suppose the energy hessian **(9)** of a given solution

(A,Φ) of **(•)** is positive definite (resp. semipositive).

Then the bundle $(V, \bar{\partial}_A)$ is stable (resp. semistable).

Proof.

If $(V,\bar{\partial}_A)$ is not (semi)-stable, then $\exists\, \mathbf{p} \leq V$ $\bar{\partial}_A$-holomorphic subbundle, with $\sigma(\mathbf{p}) \leq 0$ (resp. <0). Let us take a variation $u = ip \in \mathfrak{g}$, with p projection onto $\mathbf{p}$. Then we have:

$H^2(p,p) \leq 0$, so that:

$H(ip,ip) = 2\, H^2(p,p) + 4\pi\, \sigma(p) \leq 0$ (resp. <0)

□

Corollary 11. Let (A,Φ) be a solution of **(•)** on $V \longrightarrow M^2$; let $\mathbf{p} \subseteq V$ be a $\bar{\partial}_A$-holomorphic, Φ_z-invariant subbundle (i.e. a uniton), and let $p: V \longrightarrow \mathbf{p}$ be the associated projection.

Then $u = ip$ is critical point for the functional **(9)**:

$H(ip,ip) = |\, d_A(p)\,|^2 - |\,[\Phi, p\,]\,|^2$

restricted to the space of Grassmannian variations.

Proof.

Because of proposition 9, it's sufficient to show that p is a critical point of:

$H^2(p,p) = 2(\, |p^{\perp}\bar{\partial}_A p|^2 - |p^{\perp}\Phi_z p|^2\,)$

on the space of p's.

The Euler-Lagrange equations for p (we allow variations of the form $p \longmapsto g_t^{-1} p g_t$, with $g_t: (-\varepsilon,\varepsilon) \longrightarrow \mathfrak{G}$) are of the form:

$D'(p) + D''(p) = 0$ **(20)**

where $D'(p)$, $D''(p)$, are differential expressions in p, coming out from the 1st and 2nd term in $H^2(p,p)$.

But $D'(p) = 0$, because $p^{\perp}\bar{\partial}_A p = 0$, so p is a minimum of $|p^{\perp}\bar{\partial}_A p|^2$, and therefore a critical point for $|p^{\perp}\partial_A p|^2$, in the space of $\mathbf{p}$'s.

Similarly, $D''(p) = 0$, because $\mathbf{p}$ is Φ_z-invariant. Therefore, if $\mathbf{p}$ is a uniton, it satisfies the Euler-Lagrange equations **(20)**. □

Remark. Corollary 11 still holds if $\mathfrak{p}$ is an "antiuniton", i.e. a ∂_A-antiholomorphic, $\Phi_{\bar{z}}$-invariant subbundle of V. Indeed, just repeat the proof above, considering now H^1; or observe that $\mathfrak{p}$ is a antiuniton if and only if $\mathfrak{p}^\perp$ is a uniton; and that $H(i_p, i_p)=H(i_{p^\perp}, i_{p^\perp})$ for each p. □

§7 Some open questions

Question 1. *Is the converse of corollary 10 true?*

This question is closely related to the following:

Question 2. *Is the converse of corollary 11 true; in other words: is every critical point of the functional* **(9)** *on the space of complex subbundles of V either a uniton or a antiuniton?*

Question 3. *Is it possible to generalize some of the constructions and of the results in this paper to the case when M is a Kahler manifold ?*

Partial answer. Let M be a Kähler manifold, ω the Kähler 2-form, J the real endomorphism of the tangent space induced by the complex structure on M.

It is quite easy to generalise formula **(14)**, and hence Corollaries 10, 11, using the algebraic "trace operator" on 2-forms described by Donaldson in [D], and substituting connections with constant central curvature with Einstein connections on a given bundle $V \longrightarrow M$.

Analogously, it's quite easy to build up "Uhlenbeck loops" of Einstein connections, using the operator J, instead of the Hodge $*$-operator on 1-forms, in **(6)**. But the problem now is that a unitary connection A on $V \longrightarrow M$ must satisfy the integrability condition: $\bar{\partial}_A \circ \bar{\partial}_A = 0$, in order to define a genuine complex structure on V; and this condition does not intertwine well with the loop construction.

Flag transforms generalise to this context: but, as a careful reader will have noticed (cf. the proof of proposition 5), they are more closely associated with the algebraic structure of the tangent space to the space of connections, than with the proper equations **(•)** we were considering. Anyway, we were unable to prove any generalisation of theorem 8.

Similar problems sussist if we substitute the harmonicity equations with "pluriharmonicity equations" in the sense of Ohnita [O].

Question 4. Motivated by a theorem of Gaveau (cf.[G]), we ask if the following is true.

Let $V \longrightarrow M^2$ be a complex vector bundle on a compact Riemann surface M^2; and let (B,C) be two unitary connections on V, with constant central curvature $F(B) = F(C) = -2\pi i * \mu(V)$.

Then there exists a unitary connection $B^{\sim}$, gauge equivalent to B, such that $B^{\sim}$, C, are harmonic one with respect to the other.

This would give information on the moduli space of solutions of **(•)**.

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SOME REMARKS ON GEODESICS IN GAUGE GROUPS AND HARMONIC MAPS.

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SOME REMARKS ON GEODESICS IN GAUGE GROUPS AND HARMONIC MAPS.

Giorgio Valli

Table of contents.

<u>Summary</u>.

Given $P \to M$, a smooth principal SU(N)-bundle over a compact Riemannian manifold M, we consider the "gauge group" $\mathcal{G}$ of smooth automorphisms of P.

Every connection A on $P \to M$ induces a right invariant Riemannian pseudo-metric on $\mathcal{G}$, via right translation of the inner product:

$$\langle u, v \rangle = \int_M (d_A u, d_A v) \tag{0.1}$$

on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$.

In § 2, 3 we write down the Euler equations of geodesic motion in $\mathcal{G}$; in § 6 we prove local existence and uniqueness of solutions for the associated Cauchy problem, in the hypothesis of irreducibility of A; in §4 we study certain conserved quantities.

In § 7 we define " harmonic elements " of $\mathcal{G}$, with respect to the fixed connection A: they generalise harmonic maps $M \to U(N)$.

We then give two families of examples of closed geodesics in $\mathcal{G}$, connecting certain classes of such harmonic elements to the identity.

The first one consists of one-parameter subgroups of $\mathcal{G}$, associated to "harmonic subbundles" of the vector bundle $V \to M$, canonically associated to $P \to M$.

The second family is produced via a loop of connections with constant central curvature, in the case when M is a compact Riemann surface (cf. §8). (In particular we prove that Uhlenbeck's "extended

solution" (cf. [20]) is a geodesic in the space of maps $M^2 \longrightarrow U(N)$). This loop of connections is related to the theory of complete integrability, and Lax pairs with a complex parameter: cf. [10], [20], [22], [23].

Finally, again in the case when M is a Riemann surface, we show how to extract paths of holomorphic differentials and line bundles from geodesic paths in $\mathfrak{G}$.

We believe that, excepted § 8,9, most of the material is just an application of a standard scheme to the case of the gauge group acting on connections: we could have therefore considered as well, for example, the case of the group of diffeomorphisms of a manifold, with its induced action on the space of Riemannian metrics, as in the theory of elasticity. In particular, the reader may find overlaps with [17], [18].

We have tried to avoid the use of infinite dimensional differential geometry: everything here is smooth, unless otherwise stated. The reader may find details on the "Lie group" structure of $\mathfrak{G}$ in [3], [4], [15], [16], [18](for example).

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1. Introduction and dictionary.

Let M be a compact Riemannian manifold, and let $P \longrightarrow M$ be an SU(N)-

principal bundle (U(N) in §7,8,9).

Let $\mathcal{G}$ be the "gauge group" of smooth automorphisms of $P \to M$; if $V \to M$ is the complex hermitian vector bundle, canonically associated to $P \to M$ via the standard representation of SU(N) in $\mathbb{C}^N$, then we have:

$\mathcal{G} \cong$ { smooth special unitary sections of $\mathrm{End}(V) \to M$ } (1.1)

The "Lie algebra" of $\mathcal{G}$, in a heuristic sense, but which can be made rigorous (cf. also §6), is :

$\mathfrak{g}$ = { smooth sections of the bundle $\mathrm{ad}(P) \to M$ } (1.2)

(where $\mathrm{ad}(P) \to M$ is the Lie algebra bundle associated to $P \to M$ via the adjoint representation); and we may identify:

$\mathfrak{g} \cong$ { smooth $\mathfrak{su}(N)$ - sections of $\mathrm{End}(V) \to M$ } (1.3)

Let $\mathcal{A} = \mathcal{A}(P)$ be the space of connections on the principal bundle $P \to M$; $\mathcal{A}(P)$ is an affine space , modelled on the vector space $\mathfrak{g} \otimes T^*(M)$ of 1-forms, with coefficients in $\mathfrak{su}(N)$.

Each connection $A \in \mathcal{A}(P)$ induces, for each integer $k \geq 0$, an exterior differential:

$$d_A\colon\ \mathfrak{g} \otimes \Lambda^k T^*(M) \longrightarrow \mathfrak{g} \otimes \Lambda^{k+1} T^*(M) \qquad (1.4)$$

We remark that:

$$d_A d_A(\omega) = [\, K(A), \omega \,] \qquad (1.5)$$

where $K(A) \in \mathfrak{g} \otimes \Lambda^2 T^*(M)$ is the curvature of the connection A. Using local coordinates, we may identify a connection A with a 1-form $\tilde{A}$; we then have:

$$K(A) = d\tilde{A} + 1/2\, [\tilde{A}, \tilde{A}] \qquad (1.6)$$

Using the metric on M, and on $V \to M$, and the Killing form on $\mathfrak{su}(N)$, we may define Riemannian products (,) on each $\mathfrak{g} \otimes \bigwedge^k T^*(M)$; consequently, we may define operators:

$d_A^* : \mathfrak{g} \otimes \bigwedge^{k+1} T^*(M) \longrightarrow \mathfrak{g} \otimes \bigwedge^k T^*(M)$

which are adjoint to the d_A's, with respect to (,).

We can now define a "rough Laplacian" Δ_A :

$$\Delta_A = d_A^* d_A + d_A d_A^* : \mathfrak{g} \otimes \bigwedge^k T^*(M) \to \mathfrak{g} \otimes \bigwedge^k T^*(M) \qquad (1.7)$$

for each $k \geq 0$ (but we will use it only for $k = 0$).

The action of $\mathcal{G}$ on $P \to M$ induces an action:

$\mathcal{G} \times \mathcal{A} \longrightarrow \mathcal{A} \qquad (g, A) \longmapsto g_*(A)$

defined by:

$$d_{g_*(A)} v = g^{-1} d_A(gvg^{-1})\, g \qquad (1.8)$$

for each $v \in \mathfrak{g}$, and $g \in \mathcal{G}$ (but (1.8) is valid for $v \in \mathfrak{g} \otimes \bigwedge^k T^*(M)$).

In local coordinates, if A is represented by the 1-form $A^\sim$, then we have: $\quad g_*(A)^\sim = g^{-1} \tilde{A} g + g^{-1} dg.$

For $A \in \mathcal{A}(P)$, we indicate by $\mathcal{G}.A$ the gauge orbit of A.

2. Euler's equation for geodesics in $\mathcal{G}$.

We now fix a connection $A \in \mathcal{A}(P)$ on $P \to M$.

We consider the positive semidefinite inner product on $\mathfrak{g}$:

$$< u, v > = \int_M (d_A u, d_A v) = -\int_M \mathrm{Tr}\,(\Delta_A u . v) \qquad (2.0)$$

We equip $\mathcal{G}$ with the right-invariant pseudo-Riemannian metric induced from (2.0), via right translation; and we want to study the equations for geodesics in $\mathcal{G}$, with respect to this metric.

(We remark that this Riemannian metric on $\mathcal{G}$ is positive definite if and only if the connection A is irreducible: otherwise it's only semipositive definite, and it's called a "pseudometric". Moreover, it is not bi-invariant).

Proposition 2.1. Let g_t: $[a,b] \longrightarrow \mathcal{G}$ be a path in $\mathcal{G}$. Then the following statements are equivalent:

(i) g_t is a geodesic:

(ii) $$d/_{dt}\, \Delta_A F + [\, \Delta_A F, F\,] = 0 \qquad \forall t \in [a,b] \tag{2.1}$$
(where $F = g'g^{-1}$).

Proof.

Let us consider the energy functional:

$$L^{\sim}(g) = 1/2 \int_a^b \langle g'g^{-1}, g'g^{-1} \rangle \, dt \; = \int_a^b dt \int_M (d_A(g'g^{-1}), d_A(g'g^{-1}))$$

on the space of smooth maps g: $[a,b] \longrightarrow \mathcal{G}$ (smooth in a elementary sense: locally, on $U \subseteq M$, the map $g : [a,b] \times U \longrightarrow SU(N)$ must be smooth).

We consider a 1-parameter variation:

$g = g(s,t) : (-\varepsilon,\varepsilon) \times [a,b] \longrightarrow \mathcal{G}$

$g(0,t) = g(t) \;\; \forall t; \quad g(s,a) = g(a) \; g(s,b) = g(b) \;\; \forall s;$

and we compute the first variation of $L^{\sim}$.

$$d/_{ds}\, L^{\sim}(g)_{|s=0} = \int_a^b dt \int_M (d/_{ds} d_A(g'g^{-1}), d_A(g'g^{-1})\,)_{|s=0} =$$

$$= \int_a^b dt \int_M (d/_{ds}(g'g^{-1})_{|s=0},\; \Delta_A(g'g^{-1})\,)$$

Let $d/_{ds}\,(g)_{|s=0}\, g^{-1} = v$.

$$d/_{ds}\, L^{\sim}(g)_{|s=0} = \int_a^b dt \int_M (d/_{dt}(v) + [v, g'g^{-1}], \Delta_A(g'g^{-1})]) =$$

$$= \int_a^b dt \int_M (v, -d/_{dt}\Delta_A(g'g^{-1}) + [\, g'g^{-1}, \Delta_A(g'g^{-1})\,]\,)$$

Therefore $d/_{ds}\, L^{\sim}(g)_{|s=0} = 0$ for each variation $g^{\cdot}g^{-1} = v$ if and only if the right hand side satisfies the equation (2.1). □

We call (2.1) <u>(the first) Euler equation.</u>

<u>**Remarks.**</u>

Given F: $[a,b] \rightarrow \mathfrak{g}$, satisfying equation (2.1), we can recover the original map $g:[a,b] \rightarrow \mathcal{G}$, by solving, for each $x \in M$, the linear ordinary differential equation: $g' = F\, g$; so eq. (2.1) is essentially equivalent, modulo right multiplication of g_t by a constant element of $\mathcal{G}$, to the geodesic equation in $\mathcal{G}$.

Similarly, the Cauchy problem:

$g: [a,b] \rightarrow \mathcal{G}$ geodesic

$g(a) = g_a\,,\ \ g'(a) = g'_a$

is equivalent to the Cauchy problem in $\mathfrak{g}$:

$d/_{dt}\, \Delta_A F + [\, \Delta_A F, F\,] = 0$

$F(a) = F_a, \qquad F: [a,b] \rightarrow \mathfrak{g}.$

<u>3. Euler' s equation for geodesic paths in gauge orbit of connections.</u>

We give now an equivalent formulation of eq. (2.1).

Remember we have fixed a connection A on $P \to M$, in order to give a pseudo-Riemannian structure to $\mathcal{G}$. Moreover, this choice allows us to consider the map $j: \mathcal{G} \to \mathcal{A}$ defined by $g \mapsto g_* A$. This map is an isometry, with respect to the pseudo-metric on $\mathcal{G}$, and to the natural L^2-metric on the space of connections $\mathcal{A}$.

We can use j to associate to each path $g_t:[a,b] \dashrightarrow \mathcal{G}$, a path of connections $A_t = (g_t)_* A$, in the same gauge orbit of A; and conversely, given a path of connections A_t, lying in the same gauge orbit of A, we can find, not uniquely if the connection A is reducible, a path $g_t:[a,b] \dashrightarrow \mathcal{G}$, such that $A_t = (g_t)_* A$. (This is due to a "local slice theorem" (cf. [3], [4])).

We remark that, using standard Sobolev spaces, as in §6, each $\mathcal{G}$-orbit may be considered as a submanifold of the affine space A, with tangent space:

$$T_B(\mathcal{G}.B) = \operatorname{Im} d_B : \mathfrak{g} \to \mathfrak{g} \otimes T^*(M) \qquad (3.1)$$

(cf. [3],[4]); in particular, each gauge orbit inherits a Riemannian structure.

<u>Definition 3.1</u>. We say that a path A_t of connections, lying in the same gauge orbit, is a geodesic path (in a $\mathcal{G}$-orbit), if for each t the acceleration vector A_t'' is orthogonal to the $\mathcal{G}$-action.

We remark that this is equivalent, using (3.1), to the equation:

$$d_{A_t}^* (A_t'') = 0 \quad \forall t \qquad (3.2)$$

because $\ker d_{A_t}^* = (\operatorname{Im} d_{A_t})^{\perp}$.

Definition 3.1 has a quite transparent geometrical meaning. We can easily show that a path A_t in a $\mathcal{G}$-orbit of connections is geodesic if and only if it extremizes the appropriate energy (length) functional:

$$L^{\sim}(A_t) = 1/_2 \int_a^b (A_t', A_t')\, dt = L^{\sim}(g_t)$$

(where g_t is an associated path in $\mathcal{G}$, as in prop. (3.2)). Moreover, because the map j above is an isometry, we expect geodesics in $\mathcal{G}$ and in $\mathcal{G}.\mathcal{A}$ to correspond via j. Indeed, we have the following:

<u>Proposition 3.2</u>. Let A_t be a path of connections on a SU(N)-bundle $P \longrightarrow M$, lying in the gauge orbit of a fixed connection A.

Then the following statements are equivalent:

(i) A_t is a geodesic path in the gauge orbit of A;

(ii) $A_t = (g_t)_*(A)$ where $g_t:[a,b] \longrightarrow \mathcal{G}$ is a geodesic path in $\mathcal{G}$;

(iii) $d/_{dt}\,(d_{A_t}{}^* A_t') = 0 \quad \forall t$ (3.3)

<u>Proof</u>.

(3.2) $\Longleftrightarrow$ (3.3) is a trivial consequence of the fact that, $\forall v \in \mathfrak{g} \otimes T^*(M)$, we have $[v, *v] = 0$ (where $*$ is the Hodge operator:

$$* : \mathfrak{g} \otimes \Lambda^k T_*(M) \longrightarrow \mathfrak{g} \otimes \Lambda^{\dim M - k} T^*(M) \qquad (3.4)\ ).$$

(ii) $\Longleftrightarrow$ (iii)

$A_t' = d_{A_t}(g^{-1}g') = g^{-1}(d_A(g'g^{-1}))\, g$ (by 1.8).

$d_{A_t}{}^*(A_t') = g^{-1}(d_A * d_A(g'g^{-1}))g = g^{-1}(\Delta_A(g'g^{-1}))g$ So, we have:

$d/_{dt}\, d_{A_t}{}^*(A_t') = 0 \Longleftrightarrow d/_{dt}\,(g^{-1}\Delta_A(g'g^{-1})\, g) = 0 \Longleftrightarrow$

$\Longleftrightarrow g^{-1}\,(\, d/_{dt}\, \Delta_A(g'g^{-1}) - [g'g^{-1}, \Delta_A(g'g^{-1})\,]\,)\, g = 0\,.$

□

We call equation (3.3) <u>(the second) Euler equation</u>.

4. Conserved quantities and moment map.

As in finite dimensional geometry, the geodesic flow on the tangent space to a gauge orbit $T(\mathcal{G}.A)$ is canonically dual, via the L^2 inner product (,) on $\mathfrak{g} \otimes T^*(M)$, to the hamiltonian flow on the cotangent bundle $T^*(\mathcal{G}.A)$, with hamiltonian function the riemannian squared norm.

The $\mathcal{G}$-action on the space of connections $\mathcal{A}$ induces a $\mathcal{G}$-action on the cotangent bundle $T^*(\mathcal{A})$, and hence on $T^*(\mathcal{G}.A)$:

$$(g, (A,T)) \longmapsto (g_*(A), g^{-1}T\,g)$$

This action is, by general principles in Hamiltonian mechanics, symplectic and almost hamiltonian (cf. [9], or any other book on geometric Hamiltonian mechanics).

Dualizing, we get a moment map:

$$P : T^*(\mathcal{A}) \longrightarrow \mathfrak{g}^* \cong \mathfrak{g} \qquad P(A,T) = -\,{d_A}^*(T) \qquad (4.1)$$

(we leave the computation to patient readers ; it's not difficult, but it needs some manipulations of the definitions ; cf. also a similar computation in [11]).

In particular, ${d_{A_t}}^* A_t'$ is invariant under the geodesic flow in $\mathcal{A}$, or in any $\mathcal{G}$-orbit.

Therefore, eq. (3.3) express the <u>conservation of momentum</u>.

The conservation of energy is given by the invariance of the arclength (A_t', A_t') under time evolution. Indeed we have: $d/_{dt}\,(A_t',A_t') = 2(A_t'',A_t') = 0$, because A_t' is tangent to the $\mathcal{G}$-orbit (by assumption), and A_t'' is orthogonal to it (by definition of geodesic).

We can recover part of this conservation laws in eq. (2.1); (only part of them, because eq. (2.1) has less dependant variables).

Proposition 4.1. Let $F:[a,b]\to\mathfrak{g}$ be a solution of (2.1). Then:

(i) the functions $I^k(x) = Tr(\Delta_A F)^k(x)$ $x\in M$

are constant of the motion;

(ii) the energy $E(F) = 1/_2 \int_M (\Delta_A(F),F)$

is a constant of the motion;

(iii) if g and A_t are the corresponding geodesic paths in $\mathcal{G}$, and in $\mathcal{G}.A$ ($F=g'g^{-1}$ and $A_t = (g_t)_* A$), so that, by eq. (3.2),

$d_{A_t}{}^* A_t' = C \quad \forall t,$ then we have:

$I^k(x) = Tr\,(C)^k, \quad E\,(F) = 1/2 \int_M (\Delta_A F,F) = (A_t',A_t').$

Proof.

(i) and (ii) follow from (iii). Moreover, as in the proof of prop.3.2,

$C = d_{A_t}{}^* A_t' = g^{-1}\Delta_A f\, g\,,\ A_t' = g^{-1}(d_A F)\, g$, and g is unitary.

□

Remarks.

(i) (4.1) (i) easily follows from direct computation as well, because (2.1) is in the form of a Lax pair, and this produces the "isospectral" evolution of $\Delta_A F$.

(ii) The group of real positive functions on M, acts via conformal transformations on the space of metrics on M, but it does not act on the space of connections. Therefore, it changes the geodesic equations, by changing the "Hamiltonian", but it does not produce any moment map. When dim M = 2 the action is anyway trivial (cf.§ 8).

5. Historical remarks.

(A) The study of geodesics in a Lie group, equipped with a right (or left) invariant metric, dates back to Euler's work on the motion of a rigid body with one fixed point in $\mathbb{R}^3$. Indeed, this physical problem is mathematically equivalent to studying geodesics in SO(3), equipped with a left invariant metric. This metric, obtainable through integration of the mass distribution of the rigid body, is actually the unique physical datum of the problem.

For the sake of fun, we translate the classical notations in our setting.

Let g_t be a geodesic path in $\mathcal{G}$, and let $(g_t)_*(A)$ be the corresponding geodesic path of connections. Then:

$\omega_b = g'g^{-1}$ is the velocity vector with respect to the body;

$\omega_s = g^{-1}g'$ is the velocity vector with respect to the space;

$\Delta_A : \mathfrak{g} \to \mathfrak{g}$ is the inertia operator (I);

$M_b = \Delta_A(g'g^{-1}) = \Delta_A F$ is the kinetic moment (with respect to the body);

$M_s = d_{A_t}{}^* A_t' = C$ is the kinetic moment (w.r.t. the space).

Finally, Euler's equations for the motion of a rigid body:

$$d/_{dt}\, M_b + [M_b, \omega_b] = 0 \qquad d/_{dt}\, M_s = 0$$

are our Euler's equations (2.1) and (3.3):

$$d/_{dt}\, \Delta_A F + [\, \Delta_A F, F\,] = 0 \qquad d/_{dt}\, (d_{A_t} * A_t'') = 0.$$

(B) A second classic example of the study of geodesics in a group is the case:

$$\mathcal{G} = \{\text{ volume preserving diffeomorphisms of a manifold } M\,\}$$

(M compact or with boundary); the geodesic flow in $\mathcal{G}$ describes the motion of an ideal incompressible fluid, moving in M. The corresponding equation is still due to Euler; but we actually ignore if he was conscious of the connection with his equation for the rigid body.

For a detailed analytic study of this case, see [7].

6. Existence and uniqueness of solutions for the Cauchy problem.

We want to study now the Cauchy problem:

$$\begin{cases} d/_{dt}\, \Delta_A F + [\, \Delta_A F, F\,] = 0 \\ F(0) = F_0 \qquad F: [a,b] \longrightarrow \mathfrak{g} \end{cases} \qquad (6.1)$$

All along this §, let A be an irreducible connection on the principal SU(N)-bundle P⟶M. In particular, the irreducibility of A implies that there are not covariant constant elements of $\mathfrak{g}$ (the identity transformation is always covariant constant, but does not belong to

$\mathfrak{g}$); and that the Laplacian $\Delta_A : \mathfrak{g} \to \mathfrak{g}$ is invertible.

We introduce the Sobolev spaces (cf. §1):

$$\mathfrak{g}^s = \{ L^{2,s}\text{-sections of the bundle ad } P \to M \} \qquad (6.2)$$

They are the Hilbert space completion of $\mathfrak{g}$ with respect to the scalar product:

$$(u,v)_s = \sum_{j=1,\ldots,s} ((\Delta_A)^j u, v)$$

The following statements are well known (cf. [3], [4], [15], [16]).

Lemma 6.1. Let $s > 1/2\, m$ ($m = \dim M$). Then

(i) $\mathfrak{g}^s$ is a Banach Lie algebra:

$\exists K > 0$, $\forall u, v \in \mathfrak{g}^s$, we have $[u,v] \in \mathfrak{g}^s$ and

$$|[u, v]|_s \le K(s) |u|_s |v|_s \qquad (6.3)$$

(ii) $\mathfrak{g}^s$ is the Hilbert Lie algebra of the Hilbert Lie group:

$$\mathcal{G}^s = \{ L^{2,s}\text{- automorphisms of } P \to M \} \qquad \square$$

Lemma 6.2 . Let A be an irreducible connection on the principal SU(N)-bundle $P \to M$. Then we have:

(i) $\ker (d_A : \mathfrak{g} \to \mathfrak{g} \otimes T^*(M)) = 0$

(ii) $\Delta_A = d_A^* d_A : \mathfrak{g}^s \to \mathfrak{g}^{s-2}$ is an isomorphism, $\forall s$;

and $\Delta_A^{-1} : \mathfrak{g}^s \to \mathfrak{g}^s$ is a compact (hence continuous) operator.

$\square$

Theorem 6.3. Let A be an irreducible connection on the principal SU(N)-bundle $P \to M$.

Then for each $F_0 \in \mathfrak{g}^s$, $s > 1/2\, m + 2$ ($m = \dim M$), the Cauchy problem (6.1) has a unique solution $F_t : \mathbb{R} \to \mathfrak{g}^s$.

In particular, if F_0 is smooth , then $F(t)$ is smooth for each t.

Proof.

We write the system (6.1) in a different form: set $v = \Delta_A F \in \mathfrak{g}^{s-2}$; (6.1) is then equivalent to :

$$d/_{dt}\, v + [v, \Delta_A^{-1} v] = 0 \qquad (6.4)$$

$$v(0) = v_0 \qquad v: [0,r] \longrightarrow \mathfrak{g}^{s-2}$$

We write the system (6.4) in a more abstract form:

$$d/_{dt}\, v + f(v) = 0\,,\ \ v(0) = v_0 \qquad (6.5)$$

where $f: X \longrightarrow X$ (X Banach space).

In our case we have: $X = \mathfrak{g}^{s-2}$ and $f: \mathfrak{g}^{s-2} \longrightarrow \mathfrak{g}^{s-2}$ is the non linear map:

$$f(v) = [\Delta_A^{-1} v, v] \qquad (6.6)$$

The method for solving (6.5) is exactly the same as the one normally used for solving the Cauchy problem for ordinary differential equations, starting from the transformation of (6.6) in the integral equation:

$$v(t) = v_0 + \int_0^t f(v(\tau))\, d\tau \qquad (6.7)$$

The following Proposition is a summary (with no pretension of generality) of Theorems 6.1.1 and 6.1.3 in [19], and of Theorems 5.1.1 and 5.6.1 in [13].

Proposition 6.4. Suppose that:

(i) The function f in (6.5) is of class C^1.

(ii) There exists a continuous non decreasing function $g: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that, for each $t \geq 0$, the solution of the Cauchy problem:

$$r' = g(r), \quad r(0) = r_0$$

exists for all positive times; and we have:

$$|f(u)| \leq g(|u|) \text{ for each } u \in X.$$

Then there exists a solutions $u:[0, \infty) \longrightarrow X$ to the Cauchy problem (6.5). This solution is unique.

□

It is quite easy to check, that, for each $s-2 \geq 1/2\ m$ the function f given by (6.6) satisfies the assumptions (i) and (ii) of Proposition 6.4. This is a simple consequence of its quadratic nature, and of lemmas 6.1 and 6.2. It is indeed possible to choose as function g in (ii) : $g(r) = C(s)\,r^2$. Therefore, by proposition 6.4, there exists a solution of the system 6.4 $u:[0,\infty) \longrightarrow \mathfrak{g}^{s-2}$, and hence a solution $F:[0,\infty) \longrightarrow \mathfrak{g}^{s}$ of (6.1). The existence for $t \leq 0$ is then a consequence of a simple argument of time inversion (cf. for example Remark 5.6.1 in [13]).

□

<u>Corollary 6.5</u> The gauge group $\mathfrak{G}$ is complete, in the sense of differential geometry, with respect to the pseudometric (0.1).

<u>Notes</u>.

For an eventual Hopf-Rinow type theorem, cf.[6]; but Dowling's results don' t look to be appliable to our situation.

For a detailed analytic study of a related problem (the study of geodesics in the group of volume preserving diffeomorphisms of a manifold M), cf. the article by Ebin and Marsden [7] .

Remarks.

The proof of Theorem 6.3 is quite easy. We support the idea that Euler's equation (2.1) should be considered as an ordinary differential equation, even if in a infinite dimensional space, with the Laplacian playing the role of an algebraic term, representing the curvature of $\mathcal{G}$; as it should be, being an equation describing geodesics.

If the connection A on $P \to M$ does not satisfy the conditions in lemma 6.2, so that the associated Laplacian Δ_A is not invertible, then we do not expect existence and uniqueness theorems to hold for the Cauchy System (6.1). Indeed, let us suppose we are looking for a solution of (6.1), expressed as a formal power series in t:

$F(t) = \sum_{j\geq 0} t^j F_j$ with $F_j \in \mathfrak{g}$. Then (6.1) becomes:

$$\Delta_A F_{j+1} = 1/_{j+1} \sum_{0\leq h\leq j} [F_{j-h}, \Delta_A F_h] \qquad \forall j\geq 0 \qquad (6.8)$$

To solve (6.8) we have to proceed by induction on j; but, if the conditions of lemma 6.2 are not satisfied, then the Laplacian Δ_A is neither injective or surjective; therefore, at each stage, a choice of an F_j is not always possible, or, when possible, it is not unique.

In this case, trying to solve the Cauchy problem for the second Euler equation looks more promising:

$$d/_{dt}\,(d_{A_t}{}^{*} A_t') = 0 \quad A(0) = A_0,\ A'(0) = A'_0$$
$$A'(t) \in \mathrm{Im}\,\{d_{A_t}: \mathfrak{g} \to \mathfrak{g} \otimes T^*(M)\} \qquad (6.9)$$

We indeed expect existence and uniqueness theorems to hold for (6.9); but rather than consider (6.9) directly (one could use the conservation of the momentum, to restrict the problem to integration of a vector field, but the main difficulty would be then the non- linear nature of the space on which it would be defined - a finite codimensional subspace of a gauge orbit), it may be more fruitful to restrict oneself to the study of based gauge group and gauge Lie algebras in a first place.

We are not experts in the subject, so we prefer to leave the problem open.

7. Stationary solutions and harmonic elements.

Let $P \longrightarrow M$ be an U(N)-principal bundle over a compact Riemannian manifold M; Let $A \in \mathcal{A}(P)$ be a fixed connection on it ; let $\mathcal{G}$ be the gauge group of smooth automorphisms of $P \longrightarrow M$, and let $\mathfrak{g}$ be its Lie algebra of smooth sections of $ad(P) \longrightarrow M$.

Definition. A solution $F: [a,b] \rightarrow \mathfrak{g}$ of the 1st Euler equation:

$$d/_{dt}\, \Delta_A F + [\, \Delta_A F, F\,] = 0 \qquad (2.1)$$

is called stationary if

$$d/_{dt}\, (\Delta_A F) = 0 = [F, \Delta_A F] \qquad (7.1)$$

Examples.

(i) Take F independant of time, satisfying the eq. :

$\Delta_A(F) = \lambda(x)F(x)$.

(ii) In the case of geodesics in SO(3), describing the motion of a rigid body with one fixed point in $\mathbb{R}^3$ (cf.§5), stationary solutions are just pure rotations of the body, with rotation axes the eigenvectors of the inertia operator. They are 1-parameter subgroups of SO(3).

(iii) In the case of geodesics in the group of volume preserving diffeomorphisms of a manifold M, describing the motion of an incompressible ideal fluid in M, stationary solutions are just those which preserve the velocity vector of the fluid, in each point of M.

Let now B be another connection. Then A - B is a $\mathfrak{u}(N)$-valued 1-form, and we may consider the L^2- energy:

$$E: \mathcal{A}(P) \to \mathbb{R} \quad E(B) = 1/2 \int_M (A-B, A-B)_{\mathfrak{g}\otimes T^*(M)} \qquad (7.2)$$

<u>Definitions</u>.

(i) We call B harmonic (with respect to A) if B is a critical point for E in its orbit.

(ii) We say that $g \in \mathcal{G}$ is *harmonic* (is an harmonic element, or an harmonic gauge), with respect to A, if g_*A is an harmonic connection (w.r.t. A).

<u>Proposition 7.1</u>. With the notations above:

(i) B is harmonic (w.r.t. A)

if and only if $d_A^*(A-B) = 0$ (7.3)

if and only if A is harmonic (w.r.t. B).

(ii) $g \in \mathcal{G}$ is harmonic (w.r.t. A)

if and only if $d_A^*(A - g_*A) = 0$

if and only if $d_A^*(A - (g^{-1})_*(A)) = 0$

if and only if $d_A^*(g^{-1}d_A g) = 0$ (7.4)

<u>Corollary 7.2</u>.

$$g \in \mathfrak{G} \text{ is harmonic} \iff g^{-1} \text{ is harmonic.}$$

Proof.

(i) B is harmonic with respect to A if and only if every variation B' of B, which is tangent to the gauge orbit, is orthogonal to the distance vector A-B. But the tangent space in B to the $\mathfrak{G}$-orbit is given by (3.1) Im d_B; so B is harmonic w.r.t. A if and only if $A-B \in (\mathrm{Im}\, d_B)^{\perp} = \mathrm{Ker}\, d_B{}^{*}$.

But $$d_A{}^{*}(A-B) = d_A{}^{*}(A-B) \tag{7.5}$$

because, for each $X, Y \in \mathfrak{g} \otimes T^*(M)$, we have : $[X, *Y] + [Y, *X] = 0$; (where $*$ is the usual Hodge $*$ operator).

(ii) easily follows from (i), and from formula (1.8).

□

Corollary 7.2 states that the involution:

$J: \mathfrak{G} \to \mathfrak{G} \qquad J(g) = g^{-1}$

restricts to the space of harmonic elements of $\mathfrak{G}$. The fixed point set of J is:

$$GR(\mathfrak{G}) = \{\, g \in \mathfrak{G} \mid g^2 = 1 \,\} \tag{7.6}$$

(GR stays for Grassmannian); it may be identified with the space of subbundles of the complex vector bundle $V \to M$ canonically associated to $P \to M$ via the standard representation of U(N) in $\mathbb{C}^N$. Indeed, $g^2 = 1$ if and only if we can write (uniquely) $\quad g = (p^{\perp} - p) \qquad$ (7.7)

where $p = p^2$ is the hermitian projection operator onto a subbundle $\mathfrak{p}$ of $V \to M$; and $p^{\perp} = (p^{\perp})^2 = 1-p$ is the hermitian projection operator onto the hermitian complement $\mathfrak{p}^{\perp}$ of $\mathfrak{p}$.

Definition. We say that a subbundle $\underline{p}$ of $V \to M$ is harmonic with respect to a connection A if the associated element $g = p^{\perp} - p \in GR(\mathcal{G})$ is harmonic (w.r.t. A).

Proposition 7.3. Let $g \in \mathcal{G}$, $g^2 = 1$, so that $g = (p^{\perp} - p)$. Then we have:

(1) $E(g) = 1/2\, |g_* A - A|^2 = 2 \int_M (d_A p, d_A p)$

(2) The following statements are equivalent.

(i) g is harmonic.

(ii) g is a critical point of the energy (7.2), with respect to variations in $GR(\mathcal{G})$.

(iii) $[\Delta_A p, p] = 0$ (7.8)

Proof.

(1) follows from patient computations, which we prefer to omit.

(2) By (1), (ii) is equivalent to p being a critical point of

$$E(p) = 2 \int (d_A p, d_A p)$$

in the space of p's. We allow variations of the form: $t \mapsto h^{-1} p h$, where $h:(-\varepsilon,\varepsilon) \to \mathcal{G}$, $h(0) = 1$. Therefore $p' = [\,p, v]$, with $v \in \mathfrak{g}$; and

$$E'(p) = 4 \int (p', \Delta_A p) = -4 \int (v, [p, \Delta_A p]$$

Therefore $E'(p) = 0\ \forall v \in \mathfrak{g} \iff [p, \Delta_A p] = 0$

By prop. 7.1, g is harmonic if and only if $d_A{}^*(g_* A - A) = 0$. We can use local coordinates on M, identifying the connection A, with a 1-form $\tilde{A} \in \mathfrak{g} \otimes T^*(M)$, with $d_A(.) = d(.) + [\tilde{A}, .\,]$; then we have :

$g_*(A) - A = g^{-1}\tilde{A} g + g^{-1} dg - \tilde{A}$; write $g = 1 - 2p$:

$g_*(A) - A = 2(2p\tilde{A}p - p\tilde{A} - \tilde{A}p + [p, dp]\,) = 2\,([p, dp + [\tilde{A}, p]\,])$.

$= 2(\,[p, d_A p]\,)$; and we have:

$d_A{}^*[p, d_A p] = [p, \Delta_A p]$.

□

<u>Example</u>. let $P = M \times U(N)$, so that $\mathcal{G} = \{$ smooth maps $M \to U(N)$ $\}$, and $GR(\mathcal{G}) = \{$ smooth maps $M \to G_k(\mathbb{C}^N)$, with $1 \le k \le N-1$ $\}$; and let A be the 0-connection, given by standard differentiation. Then an element in $\mathcal{G}$ (resp. in $GR(\mathcal{G})$) is harmonic if and only if it is <u>an harmonic map</u> $M \to U(N)$ (resp. $M \to G_k(\mathbb{C}^N)$). The equivalence of (i) and (ii) is nothing but the standard fact that an harmonic map into $G_k(\mathbb{C}^N)$ is harmonic if and only if it is harmonic as a map into U(N). This is consequence of the fact that the inclusion map (7.7), viewed as a map $G_k(\mathbb{C}^N) \to U(N)$ is totally geodesic.

Let $g = (p^\perp - p) \in \mathcal{G}$ define a subbundle $\underline{p} \subseteq V \to M$. We associate to $\underline{p}$ the loop:

$$g_t : [0, 2\pi] \to \mathcal{G} \quad t \mapsto (p^\perp + e^{it} p) = \exp(itp)$$

It is *<u>a one parameter subgroup</u>* of $\mathcal{G}$.

<u>Theorem 7.4.</u> Let $g \in \mathcal{G}$, $g^2 = 1$; and let $g_t = \exp(itp)$ be the associated 1-parameter subgroup of $\mathcal{G}$, as above.

Then the following statements are equivalent.

(i) $g = g_\pi$ is an harmonic element of $\mathcal{G}$.

(ii) $t \mapsto g_t$ is a (stationary) geodesic in $\mathcal{G}$, of length $\pi(2E(g))^{1/2}$.

<u>Proof.</u>

$g = g_\pi$ is harmonic $\Longleftrightarrow [\Delta_A p, p] = 0$

g_t is a geodesic $\Longleftrightarrow F = g'g^{-1}$ satisfies the first Euler equation (2.1):

$$d/_{dt}\, \Delta_A F + [\Delta_A F, F] = 0$$

$$\Longleftrightarrow F = i\, p \text{ satisfies } [\Delta_A F, F] = 0.$$

Moreover the length of g_t is :

$$L(g_t) = 2\pi\, |d_A p| = 2\pi\, (1/2\, E(g))^{1/2} = \pi\, (2E(g))^{1/2}.$$

□

Definition. The geodesic length spectrum of $\mathfrak{G}$ (with respect to the connection A) is the set of length of closed geodesics in $\mathfrak{G}$.

The energy spectrum of harmonic elements of $\mathfrak{G}$ (w.r.t. A is the set of energies of harmonic elements of $\mathfrak{G}$. In particular, the energy spectrum of harmonic subbundles of $V \longrightarrow M$ is the energy spectrum of harmonic idempotent elements g of $\mathfrak{G}$.

Corollary 7.5. The energy spectrum of harmonic subbundles of $V \longrightarrow M$ is contained in $1/2\ \pi^2$ the square of the geodesic length spectrum of $\mathfrak{G}$.

Proof.

By theorem 7.4, we can associate to each harmonic $g = (p^\perp - p)$ the closed geodesic $t \longmapsto g_t = (p^\perp + e^{it} p)$.

□

Example.

Let us consider, for simplicity, the case when M is a compact Riemann surface of genus h, $P = M \times U(N)$, and A is the zero connection. Then the construction in this § associates to each harmonic map

$g = (p^{\perp} - p): M \to G_k(\mathbb{C}^N)$ the geodesic in the group $\mathcal{G}$ of smooth maps $M \to U(N)$ $g_t = \exp(itp)$.

By a theorem of Killingback [12], $\pi_1(\mathcal{G}) = \pi_2(U(N))^{2h} \oplus \pi_3(U(N)) \cong \mathbb{Z}$. Therefore we can associate a degree $\delta(g_t)$ to each map $g_t: S^1 \to \mathcal{G}$.

More over, we can define the topological degree $d(g)$ of a map $g: M^2 \to G_k(\mathbb{C}^N)$ as the algebraic degree of the induced map in cohomology: $f^*: H^2(G_k(\mathbb{C}^N), \mathbb{Z}) \cong \mathbb{Z} \to \mathbb{Z} \cong H^2(M, \mathbb{Z})$.

<u>Proposition 7.6</u> Let $g: M^2 \to G_k(\mathbb{C}^N)$, and let g_t be the associated 1-parameter subgroup of the space $\mathcal{G}$ of maps $M^2 \to U(N)$. Then the topological degree of g is equal to the topological degree of the loop g_t.

<u>Proof</u> (sketch).

To compute the degree of g_t, we view g_t not as a loop in $\mathcal{G}$, but as a map from M^2 into the "loop group" (cf. [2]) $\Omega U(N)$. The positive generator of the second cohomology group of $\Omega U(N)$ is the left invariant, symplectic 2-form S (cf. [2], [22]) induced via left translation by:

$$S(\xi, \chi) = 1/_{8 i \pi^2} \int_{S^1} (\xi', \chi)$$

for ξ, χ in the Lie algebra of $\Omega U(N)$.

Therefore, doing some computation (cf. [21],[22]), we get :

$$\delta(g_t) = \int_M (g_t)_*(S) = 1/_{2\pi} \int_M |p^{\perp}\bar{\partial} p|^2 - |p^{\perp}\partial p|^2 = d(g)$$

$\square$

<u>Remark</u>. The constructions in this § may be generalized to other

groups and symmetric spaces, rather than U(N) and the complex Grassmannian.

As an example, let M be a compact Riemannian manifold of even dimension, and let A be a connection on its tangent space. Let $\mathfrak{G}$ be the gauge group:

$$\mathfrak{G} = \{ \text{smooth orthogonal sections of } \mathrm{Aut}(M) \longrightarrow M \}$$

We may define, via A, a right invariant metric on $\mathfrak{G}$, as in §1.

Let us consider the space:

$$\mathfrak{J} = \{ J \in \mathfrak{G} \mid J^2 = -1 \}$$

$\mathfrak{J}$ may be described as the space of almost complex structures on M, which are compatible with its given Riemannian structure.

We consider the energy $E : \mathfrak{G} \longrightarrow \mathbb{R}$, and its restriction to $\mathfrak{J}$; then we may define the spaces of harmonic elements of $\mathfrak{G}$, and of harmonic almost complex structures, as above.

We associate to each almost complex structure J on M, the 1-parameter subgroup of $\mathfrak{G}$:

$$t \longmapsto g_t = \exp(tJ) = \cos t + J \operatorname{sen} t.$$

We have then a complete analogue of theorem 7.3.

<u>Proposition 7.7</u>. The following statements are equivalent.

(i) $g_t = \exp(tJ)$ is a (stationary) geodesic in $\mathfrak{G}$.

(ii) $J = g_{\pi/2}$ is an harmonic almost complex structure.

(iii) $[\Delta_A J, J] = 0$ (7.10)

Note: eq. (7.10) was first shown to me by C. M. Wood.

8. Harmonic gauges on Riemann Surfaces & Zakharov-Shabat representation.

Let us take now $M = M^2$ compact Riemann surface, equipped with an hermitian metric, and $P \to M$ principal $U(N)$-bundle over M. Let $\mathcal{G}$ = Aut(P) be the gauge group, and $\mathfrak{g}$ its Lie algebra.

It is well-known that the unique topological invariant of a $U(N)$-principal bundle $P \to M^2$ is its first Chern class $c_1(P) \in H^2(M,\mathbb{Z}) \cong \mathbb{Z}$; or equivalently, the normalised 1st Chern class:

$$\mu(P) = c_1(P)/\text{rank } P \qquad (8.1)$$

Let $\mathcal{A} = \mathcal{A}(P)$ be the space of connections on $P \to M$. We consider the subspace of A(P):

$$\mathcal{A}^{\mu}(P) = \{ A \in \mathcal{A}(P) \mid *K(A) = -2i\pi\, \mu(P) \} \qquad (8.2)$$

consisting of connections with constant central curvature $-2i\pi\mu(P)$.

The space $\mathcal{A}^{\mu}$ is not empty, because by a well known theorem of Narasimhan and Seshadri, it contains, in particular, stable bundles (cf.[3], [5]). Its "tangent space" at each point $B \in \mathcal{A}^{\mu}$ is given by:

$$T_B(\mathcal{A}^{\mu}) = \{ \ker d_B : \mathfrak{g} \otimes T^*(M) \to \mathfrak{g} \otimes \Lambda^2 T^*(M) \} \qquad (8.3)$$

We remark that $\mathcal{A}^{\mu}$ is stable under the action of $\mathcal{G}$. Infinitesimally, this is expressed by comparing (8.3) and (3.1): indeed, for each $B \in \mathcal{A}^{\mu}$ we get, from (1.5) and the constant central curvature condition for the connection B, we get:

$$d_{\underline{B}} \circ d_{\underline{B}} = 0 \qquad (8.4)$$

so that:

$$T_B(\mathcal{A}^{\mu}) = \ker d_B \supseteq \operatorname{Im} d_B = T_B(\mathcal{G}.B)$$

For the sake of simplicity, if $B \in \mathcal{A}^{\mu}$, we say that B is a connection with μ-curvature.

As usual, we fix now a connection A on $P \longrightarrow M$; moreover, we take A with μ-curvature. As in §7, we may form the L^2-energy:

$E: A \longrightarrow \mathbb{R}$

$$E(B) = 1/2\, |A-B|^2 = 1/2 \int_M \operatorname{Tr}(A-B) \wedge *(A-B) \qquad (8.5)$$

<u>Remarks</u>.

Because of the conformal invariance of the Hodge $*$-operator on 1-forms in dimension 2, E is invariant with respect with conformal changes of metric on M. Similarly, the usual right invariant metric on the gauge group $\mathcal{G}$, given by right translation of (0.1), is conformally invariant, and therefore the equations for geodesics in $\mathcal{G}$. What is not conformally invariant is the condition for a connection to have <u>constant</u> curvature, since it needs a volume form on M, in order to make sense. Anyway, once we have chosen the space $\mathcal{A}^{\mu}$, we will need only a conformal class of metrics on M^2, i.e. a complex structure, in the following.

As in §7, we define the spaces of harmonic connections, and of harmonic gauges (always with respect to the fixed μ-curvature connection A).

<u>Proposition 8.1</u>. Let B be a connection with μ-curvature on $P \longrightarrow M^2$. Let us denote $\Phi = 1/2\,(A-B) \in \mathfrak{g} \otimes T^*(M)$, $Q = 1/2\,(A+B) \in \mathcal{A}(P)$.

Then the following statements are equivalent:

(i) B is harmonic (w.r.t. A).

(ii) for each $t \in [0,2\pi]$ the connection:

$$A_t = Q + \cos t\, \Phi + \text{sen } t *\Phi \quad A_0 = A,\ A_\pi = B \qquad (8.6)$$

has μ-curvature.

(iii) for each $t \in [0,2\pi]$ $A_{t+\pi}$ is harmonic with respect to A_t.

(iv) (Q, Φ) satisfies the following system of equations.

$$\begin{cases} *(K(Q) + 1/2\,[\Phi,\Phi]) = -2i\pi\, \mu(P) \\ d_Q\Phi = d_Q(*\Phi) = 0 \end{cases} \qquad (8.7)$$

<u>Proof</u>.

We know, by prop. 7.1, that B is harmonic w.r.t. A if and only if $d_A*(A-B) = 0$. This is equivalent to $d_Q(*\Phi) = 0$. But we have:

$K(A_t) = 1/2\,(1+\cos t)K(A) + 1/2\,(1-\cos t)K(B) + 1/2 \cos t\, d_B*(A-B)$

Therefore $*K(A_t) = -2\pi i\mu \iff d_B*(A-B) = 0$, $*K(A) = *K(B) = -2i\pi\mu$

Moreover $*K(A) = *K(B) = -2i\pi\mu$ is equivalent to:

$*(K(Q) + 1/2\,[\Phi,\Phi]) = -2i\pi\mu(P)$, $\quad d_Q\Phi = 0$

Moreover, the invariance of the system (8.7) under the S^1-action:

$\Phi \longmapsto \cos t\, \Phi + \text{sen } t *\Phi$

implies that we could have chosen as A, B any pair of connections A_t, $A_{t+\pi}$.

□

Remarks.

We have succeeded in representing the non-linear system (8.7) in the form of μ-curvature condition for a loop of connections A_t, or, in other words, in the form of a (local) compatibility condition for a loop of linear systems:

$$d_{A_t} v_t = 0 \qquad v:[0,2\pi] \to \mathbb{C}$$

This kind of representation is usually called Zakharov- Shabat (cf. [23]), or 0-curvature representation of the non linear system (8.7), and connects its study with the theory of completely integrable system, and soliton-type equations (cf. [23]). The representation is fundamental in analysing the solutions of (8.7) .

For example, in the particular case, $P = M \times U(N)$, $\mu = 0$, $A = 0$, the loop of connections ((8.6) has been used to give complete geometrical descriptions of harmonic maps $S^2 \to U(N)$ (cf.[20],[22]) and $T^2 \to SU(2)$ (cf.[10]).

(2) The Hodge operator $*$ verifies $(*)^2 = -1$ on 1-forms over M; it defines in this way a complex structure on the space of connections. The loop (8.6) is then a circle, of centre Q and ray Φ, lying in a complex affine subspace of $\mathcal{A}$ (P) of complex dimension 1. Conversely, given any such circle, lying in the space of μ-curvature connections, then its centre Q and any one of its rays Φ, are a solutions of (8.7).

The following definition has quite a transparent geometrical meaning, and it can be given the right variational justification, exactly as for def. 3.1.

Definition. We say that a path A_t of μ-curvature connections on

$P \to M$ is a geodesic path (in the space $\mathcal{A}^{\mu}$ of μ-curvature connections) if for each t the acceleration vector A_t'' is orthogonal to $\mathcal{A}^{\mu}$.

Because of (8.4), this is equivalent to saying that, for each t, A_t'' lies in: $\quad \operatorname{Im} d_{A_t}: \mathfrak{g} \otimes \Lambda^2 T^*(M) \to \mathfrak{g} \otimes T^*(M)$.

Proposition 8.2. Let (Q, Φ) be solutions of (8.7) on $P \to M^2$. Let A_t be the associated loop of μ-curvature connections (8.6).

Then the following statements are equivalent.

(i) A_t is a geodesic loop in the space of μ-curvature connections.

(ii) A_t lies in the same $\mathcal{G}$-orbit for each t.

(iii) A_t is a geodesic loop in a $\mathcal{G}$-orbit of connections.

(iv) $\exists\, g_t : [0,2\pi] \to \mathcal{G}$ geodesic loop such that

$$(g_t)_*(A_0) = A_t \qquad \text{for each } t. \tag{8.8}$$

Proof.

(iii) and (iv) are equivalent, as noticed in §3; moreover (iii) obviously implies (ii), which in turn is equivalent to :

for each t the velocity vector A'_t is tangent to the $\mathcal{G}$-action: i.e. for each t, $A'_t \in \operatorname{Im} d_{A_t}: \mathfrak{g} \to \mathfrak{g} \otimes T^*(M)$.

But $A'_t = * A''_t$, so (ii) is equivalent to (i). Moreover (i) implies :

(A_t remains in the same $\mathcal{G}$-orbit for each t) + ($A''_t \in \operatorname{Im} d_{A_t} \subseteq \operatorname{Ker} d_{A_t}$), which implies (iii).

□

Remarks.

(1) The "momentum" (cf. §4,5) of the loop of connections A_t is:

$$d_{A_t}{*}(A_t') = -1/4\ [A-B, A-B] = -[\Phi, \Phi].$$

(2) In the case M is the Riemann sphere, the loop G_t in (8.8) always exists. This is due to the fact that on $\mathbb{CP}^1$, the space $\mathcal{A}^{\mu}$ of connections with μ-curvature always consists of a single $\mathcal{G}$-orbit (cf. [22]).

(3) Let us consider the case when $P = M^2 \times U(N)$, $\mu = 0$, $A = 0$; $\mathcal{G}$ is then isomorphic to the space of smooth maps from M^2 into U(N). Then we can look at the loop (8.8) $g_t : S^1 \to \mathcal{G}$ as a map $G: M^2 \to \Omega U(N)$, where $\Omega U(N)$ is the "loop group" of U(N) (cf[2]). The map G has been called by Uhlenbeck [20] "extended solution"; it is a holomorphic map into the infinite dimensional Kahler manifold $\Omega U(N)$. Its degree is (cf.[22]) $1/8\pi$ the energy of the harmonic map $g_\pi : M^2 \to U(N)$

(4) Killingback has computed in [12] the homotopy groups of the gauge groups $\mathcal{G} = \{$ smooth maps $M^2 \to U(N)\}$(cf. also Prop. 7.6).

(5) The length of the loop A_t in (8.6) is (cf. theorem 7.4):

$$L(A_t) = \int_0^{2\pi} (A', A')^{1/2} = \pi\,|A-B| = \pi\,(2E_A(B))^{1/2}.$$

(6) A theorem ensuring the existence of harmonic connections in 0-curvature orbits, has been given by Gaveau (cf.[8]).

(7) We remark that the eventual existence of the loop G_t (8.8) implies the Morse instability of each connection $A_{t+\pi}$, harmonic with respect to A_t; (cf.[22] for a computation of the energy Hessian).

9. Geodesics in gauge groups over Riemann surfaces produce holomorphic data.

Let, like in the previous §, $P \to M^2$ be a principal U(N)-bundle over a compact Riemann surface $M = M^2$, A a fixed connection with μ-curvature on P, $\mathcal{G}$ the gauge group of automorphisms of P.

Let $g_t:[a,b] \to \mathcal{G}$ be a geodesic path in $\mathcal{G}$. We have shown the geodesic condition to be equivalent to the 2nd Euler equation:

$$d_{A_t}*A''_t = 0 \quad \forall t \tag{3.2}$$

where $A_t = (g_t)_* A$.

Equation (3.2) means that the 1-form $*A''_t$ is d_{A_t}-closed, for each t. But A_t has μ-curvature for each t, so that $d_{A_t} \circ d_{A^t} = 0$, and we can perform an Hodge decomposition:

$$\begin{cases} *A''_t = d_{A_t} v_t + H_t & (9.1) \\ d_{A_t} H_t = 0 \quad d_{A_t} * H_t = 0 & (9.2) \end{cases}$$

where v_t and H_t are paths in $\mathfrak{g}$ and $\mathfrak{g} \otimes T^*(M)$, respectively. Applying now to (9.2) the standard decomposition of $T^*(M) \otimes \mathbb{C}$ in (1,0) and (0,1) parts, relative to the complex structure on M, we get:

$H_t = H_{z,t} + H_{\bar{z},t} \quad d_{A_t} = \bar{\partial}_{A_t} + \partial_{A_t}$ and:

$$\bar{\partial}_{A_t} H_t = 0 \tag{9.3}$$

If $\mu = 0$ and the connection A is trivial $(A = j^{-1}dj)$, then we may easily get from (9.3) a path of holomorphic 1-forms:

$$\Omega_{z,t} = (jg)^{-1} H_{z,t} (jg) \quad \bar{\partial}\Omega_{z,t} = 0$$

If this is not the case, then the standard trick, in order to extract interesting objects from (9.3), is to apply U(N)-invariant polynomials to $H_{z,t}$ (cf. [11], or the Chern-Weil theory of characteristic classes).

For a proof of the following Proposition, cf. [11].

<u>Proposition 9.1</u>. Let g_t be a geodesic path in $\mathcal{G}$; and let $H_{z,t}$ be the path in $\mathfrak{g}\otimes T^*(M)$ obtained by the above procedure. Then :

(i) for each $k\geq 1$, the coefficient of λ^{N-k} in $\det(H_{z,t}-\lambda I)$ is a path of holomorphic k-differentials on M;

(ii) generically, the space

$$M_t^{\sim} = \{ (z,\lambda) \in T^*_{1,0}(M) \mid \det(H_{z,t} - \lambda I) = 0 \}$$

is a k-fold, t-dependant branched covering of M; and the eigenspace bundle $L_t = \mathrm{Ker}(H_{z,t}-\lambda I) \subseteq V \longrightarrow M$ is a path of holomorphic line bundles over $M_t^{\sim}$.

<u>Remarks</u>. It is possible to reverse the whole procedure, finding back $H_{z,t}$ from $M_t^{\sim}$ and L_t (cf. [5] , [11], and [22]).

This construction does not appear anyway to be useful in the description of geodesics in gauge groups over Riemann surfaces: its geometrical meaning is in fact obscure, and moreover the Cauchy problem (6.9) does not translate into a Cauchy problem for holomorphic differentials or holomorphic line bundles. We believe, anyway, that there are more things to say on the subject. As a very limited attempt to formulate questions, cf. next §.

<u>10. Some problems and ideas</u>.

(1) It is well known that the motion of a rigid body with one fixed point is a completely integrable Hamiltonian system. More generally, the geodesic flow on a finite dimensional semisimple Lie group, endowed with a right-invariant Riemannian metric, is completely integrable if and only if the associated Euler equation:

$$M' = [\omega, M]$$

(which is in the form of a Lax pair), may be written in the stronger form of a "Lax pair with spectral parameter":

$$M'_\lambda = [\,\omega_\lambda, M_\lambda] \qquad \lambda \in \mathbb{C}$$

If so, all the eigenvalues of M_λ, for each λ, would provide conserved quantities for the motion, in sufficient number to ensure complete integrability (cf.[14]).

We ask the following problem: when is the geodesic flow in gauge groups a completely integrable infinite dimensional Hamiltonian system? The case of gauge groups of bundles over Riemann surfaces, as in § 8, 9, is the one when the answer is more likely to be positive, because of the conformal invariance of the equations, and of an already large amount of recent results in related subjects. We have tried, but without any success, to construct new integrals of the motion, in addition to the ones described in §4, in the form of spectral functions of matrices or of differential operators.

(2) (Again in the case of Riemann surfaces).

Uhlenbeck has described (in [20]) a decomposition of an extended solution (8.8), which in §8 we proved to be a geodesic of $\mathcal{G}$, in terms of 1-parameter subgroups of $\mathcal{G}$, satisfying certain holomorphicity conditions.

It is possible to perform a similar procedure, using some sort of iteration of the construction in §7, for some class of closed geodesics in $\mathcal{G}$?

(3) Is it possible to describe in a simple way, for example in terms of the geodesic length spectrum and of the Laplace-spectrum of $P \to M$, the geodesic length spectrum of $\mathcal{G}$? (cf. Corollary 7.5 and [21]).

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Final Chapter.

Considerations, Complements and assorted results.

1. The loop group $\Omega U(N)$.

The following facts are standard, and may be found in (for example) [A], [E-L].

We consider the space:

$\Omega U(N) = \left\{\gamma: S^1 \to U(N) \mid \gamma \text{ is } C^\infty, \text{ and } \gamma(1) = I\right\}$; and its "Lie algebra":

$\Omega \mathfrak{u}(N) = \left\{\eta: S^1 \to \mathfrak{u}(N) \mid \eta \text{ is } C^\infty, \text{ and } \eta(1) = 0\right\}$

If u, v are elements of $\Omega\mathfrak{u}(N)$, then there exist a Fourier expansion:

$u = \sum_{\alpha\in \mathbb{Z}_*} (1-\lambda^\alpha)\, u_\alpha, \quad v = \sum_{\alpha\in \mathbb{Z}_*} (1-\lambda^\alpha)\, v_\alpha$

and we introduce on $\Omega\mathfrak{u}(N)$ the inner product:

$$(u,v) = \sum_{\alpha\in \mathbb{Z}_*} |\alpha|\,(u_\alpha, v_\alpha) \tag{1.1}$$

We give $\Omega^\infty U(N)$ the Riemannian metric induced, via left translation, from (1,1).

The following facts are well known.

(i) The space $\Omega U(N)$ is an infinite dimensional Frechet manifold; moreover, with respect to ordinary pointwise multiplication, it is a Banach Lie group, with Banach Lie algebra $\Omega\mathfrak{u}(N)$.

(ii) $\Omega U(N)$ has a natural integrable almost complex structure, induced via left translation by the following anti-involution on $\Omega\mathfrak{u}(N)$

$$J(u) = i \sum_{\alpha\in \mathbb{Z}_*} (-1)^{\operatorname{sgn}\alpha}\,(1-\lambda^\alpha)\, u_\alpha \tag{1.2}$$

if $u = \sum_{\alpha\in \mathbb{Z}_*} (1-\lambda^\alpha)\, u_\alpha$

(iii) The complex structure and the metric induced by (1.1) and (1.2) give $\Omega U(N)$ the structure of an infinite dimensional Kähler manifold.

The associated Kähler 2-form, normalised so as to be the positive generator of $H^2(\Omega U(N),\mathbb{Z}) \cong \mathbb{Z}$ is given by left translation of the 2-form on $\Omega\mathfrak{u}(N)$:

$$S(\eta,\zeta) = 1/_{8\pi^2}\int_{S^1} (\eta', \zeta)\, dt \qquad (1.3)$$

(iv) If M is a complex manifold, then a map f: $M \longrightarrow \Omega U(N)$ is holomorphic if and only if the 1-form $A_z = f^{-1}\partial f$ has only positive terms in the Fourier expansion:

$$A_z = \sum_{\alpha>0} (1-\lambda^{\alpha})A_{\alpha}$$

(v) If M is (for simplicity) a compact Riemann surface, and f is a (sufficiently regular) map f: $M \longrightarrow \Omega U(N)$, then the topological degree of f, defined as the algebraic degree of the induced map $f^*: H^2(\Omega U(N),\mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathbb{Z} \cong H^2(M,\mathbb{Z})$, is given by:

$$\deg f = \int_M f^*(S) = 1/_{2\pi} \sum_{\alpha\neq0} \alpha\, |A_\alpha|^2$$

where $f^{-1}\partial f = \sum_{\alpha\neq0} (1-\lambda^{\alpha})A_{\alpha}$

The energy of f is given by:

$$E(f) = 1/_{4\pi} \sum_{\alpha\neq0} |\alpha|\, |A_\alpha|^2$$

In particular, if we define:

$$E'(f) = 1/_{2\pi} \sum_{\alpha>0} \alpha\, |A_\alpha|^2, \quad \text{and} \quad E''(f) = 1/_{2\pi} \sum_{\alpha<0} (-\alpha)\, |A_\alpha|^2,$$

then we have:

$$E'(f) + E''(f) = 2\,E(f)$$

$$E'(f) - E''(f) = \deg f$$

This is an (infinite dimensional) special case of a (finite dimensional) theorem of Lichnerowicz (cf. [E-L 1]).

2. Generalized harmonic maps and extended solutions.

Let now $V \rightarrow S^2$ be an hermitian vector bundle over S^2, endowed with the standard Riemannian metric. We follow here the notations of [V3]. We suppose $\mu(V) = c_1(V)/\mathrm{rk}\, V \in \mathbb{Z}$.

Lemma 2.1. Let A, B be connections with constant central curvature on $V \rightarrow S^2$: $*F(A) = *F(B) = -2i\pi\,\mu(V)$. Then we have:

(i) A and B are connected by a complex gauge transformation $G \in \mathcal{G}^{\mathbb{C}}$;

(ii) if A and B are unitary, then the gauge transformation is unitary $G \in \mathcal{G}$.

Proof.

We consider on the vector bundle of linear maps $\mathrm{Hom}(V_A, V_B)$ the induced connection, with covariant derivative $d_{B,A}$:

$d_{B,A}\,\Phi = d\Phi + B\Phi - \Phi A$ (in local coordinates). Then we have:

$d_{B,A}\, d_{B,A}\, \Phi = F(B)\,\Phi - \Phi\, F(A) = 0$

Therefore, on $\mathrm{Hom}(V_A, V_B)$, the connection induced from A and B is flat.

Because S^2 is simply connected, for each $p \in S^2$, there exists then one and only one solution to the system:

$$\begin{cases} d_{B,A}\,\Phi = 0 \\ \Phi(p) = I \end{cases} \qquad (2.1)$$

Analogously, we can consider the induced flat connection on the space $\mathrm{Hom}(V_B, V_A)$: then there exist one and only one solution to the system:

$$\begin{cases} d_{A,B}\,\Psi = 0 \\ \Psi(p) = I \end{cases}$$

The composed $\Phi\Psi$ is then a d_A-covariant section of $\mathrm{Hom}(V_A, V_A)$, which is the identity in p. We deduce then that $\Phi\Psi = I$, so that Φ is invertible and $\Phi_*(B) = A$.

Suppose now that both A, B are unitary connections. Then we have:

$$\begin{cases} d_B(\Phi\Phi^*) = 0 \\ \Phi\Phi^*(p) = I \end{cases} \qquad (2.2)$$

Reasoning as above, we then conclude that $\Phi\,\Phi^* = I$. Therefore the gauge transformation Φ is unitary. □

Let us consider now (as in [V3]) a solution (A, Φ) of the system of equations:

$$\begin{cases} *(F(A) + \frac{1}{2}[\Phi, \Phi]) = -2\pi i\,\mu(V) \\ \bar{\partial}_A \Phi_z = 0 \end{cases} \qquad (*)$$

where A is a unitary connection of curvature F(A), on the hermitian vector bundle $V \longrightarrow S^2$; and Φ is a 1– form with coefficients in the skew–symmetric endomorphisms of V.

As in [U], and [V3], (A,Φ) is a solution of $(*)$ if and only if the Uhlenbeck loop of connections:

$$A_\lambda = A + \lambda\,\Phi_z + \lambda^{-1}\,\Phi_{\bar{z}} \quad \lambda \in \mathbb{C}^* \qquad (2.2)$$

consists of connections with constant central curvature:

$$*F(A_\lambda) = -2\pi i\mu(V) \qquad \forall\lambda \in \mathbb{C}^*$$

The following is an immediate corollary of lemma 2.1, and mimics the construction in [U].

Proposition 2.2.

There exist a loop of complex gauge transformations:

$G_\lambda : S^1 \longrightarrow \mathcal{G}^{\mathbb{C}}$, such that we have: $\qquad G_\lambda * (A_0) = A_\lambda \qquad$ (2.3)

Moreover, G_λ is unitary for $|\lambda| = 1$.

□

Proposition 2.3. There exists a choice of the loop G_λ (2.4) such that we have a finite Fourier expansion:

$$G_\lambda = \sum_{0\le\alpha\le H} \lambda^\alpha T_\alpha \qquad \lambda\in\mathbb{C}^* \tag{2.4}$$

Proof.

The proof has been given in [V3] ; theorem 8 in [V3] is in fact equivalent to the factorization of G_λ into product of unitons:

$$G_\lambda = (p_1{}^\perp + \lambda\, p_1) \qquad (p_k{}^\perp + \lambda p_k) \tag{2.5}$$

We remark that a different proof, more analitic in character, had been previously given by Uhlenbeck in [U]., in the special case $V = S^2 \times \mathbb{C}^N$, and $A_0 = 0$. This proof consisted of 2 steps. First, it was proved that there exists an elliptic differential operator L (essentially the Jacobi operator for the energy, applied to sections of V), such that G_λ is in the kernel of L for each λ. This part is easily generalisable to the twisted case we are considering now. Then, using the finite dimensionality of the kernel of L, Uhlenbeck managed to prove the finiteness of the Fourier expansion (2.4). We think that this second step is easily generalisable to our situation, but we have not checked the details.

Let us consider for a while the situation when $V = S^2 \times \mathbb{C}^N$, $A_0 = 0$. Then G_λ is a map $G_\lambda :\ S^2 \longrightarrow \Omega U(N)$; and $G^{-1}\bar\partial G = \frac{1}{2}(1+\lambda^{-1})\, A_{\bar z}$, $G^{-1}\partial G = \frac{1}{2}(1+\lambda)\, A_z$; therefore, modulo a change of parameter $\lambda \longrightarrow -\lambda$, G_λ is a holomorphic map $S^2 \longrightarrow \Omega U(N)$ (cf.§1).

If we call $f = G_\pi$, then f is an harmonic map $S^2 \longrightarrow U(N)$; Uhlenbeck called the holomorphic map $G_\lambda :\ S^2 \longrightarrow \Omega U(N)$ extended solution (with respect to the original harmonic map f).

Moreover we have $\pi_2(\Omega U(N)) \cong \pi_3(U(N) \cong \mathbb{Z}$, so that we may compute the topological degree of the map G_λ. The following remark is not included in [V1], mainly for reason of space, but it was the origin of the energy-reduction idea in [V1] : the energy of harmonic maps $S^2 \to U(N)$ is "a priori" integral.

<u>**Proposition 2.4**</u>. Let $f : S^2 \to U(N)$ be an harmonic map; and let $G_\lambda : S^2 \to \Omega U(N)$ be its associated holomorphic map. Then we have:

$$\text{Energy}(f) = 8\pi \deg G_\lambda \tag{2.6}$$

<u>Proof</u>.

We have:

$$\deg(G_\lambda) = \int_{S^2} G_\lambda^*(S)$$

where S is the 2–form (1.3) on the loop group $\Omega U(N)$.

$$\deg(G_\lambda) = 1/8\pi^2 \int_{S^2}\int_{S^1} -4\,\mathrm{Tr}\Big(d/dt\,(G^{-1}\partial G) G^{-1}\bar{\partial} G\Big) =$$

$$= 1/8\pi^2 \int_{S^2}\int_{S^1} (-i\lambda)(1+\lambda^{-1})\,\mathrm{Tr}(A_z A_{\bar{z}}) = 1/8\pi \left(\tfrac{1}{2}\int_{S^2} |f^{-1}df|^2\right) = 1/8\pi\, E(f)$$

□

<u>**Remarks**</u>.

(i) Proposition 2.4 has been generalised by Eells and Freed to the case of harmonic maps from S^2 into any compact Lie group.

(ii) The integrality of the energy of harmonic maps $S^2 \to U(N)$ is also a consequence of Uhlenbeck's factorization theorem (2.5), as proved in [V1].

(iii) The energy is integral even in the case of solutions of the system (*) on S^2 (cf. Theorem 8 in [V3]). It' s possible to prove that an analogue of Proposition 2.4 holds: namely that the energy $2|\Phi|^2$ of a solution (A,Φ) on $V \to S^2$ is equal to 8π the degree of the loop in the gauge group (2.4) $G_\lambda : S^1 \to \mathcal{G}$.

Indeed, given any connection B with constant central curvature on V, then the 1–form on $\mathcal{G}$, defined by:

$$\omega(G,v) = \int_{M^2} \mathrm{Tr}(G^{-1} d_B G\, d_B v) \qquad \text{for } G \in \mathcal{G}, \text{ and } v \in \mathfrak{g} \tag{2.7}$$

is closed, cohomologically non-trivial, and its cohomology class does not depend from the choice of the connection B . Therefore, we may use ω, properly normalised, to compute the degree of the loop G_λ: as anticipated, the result is that this degree is $1/8\pi$ the energy of the solution (A,Φ).(cf. also some recent literature: [FA], for example).

3. Some remark on the uniton factorization

Let us return now to the general case of a solution (A,Φ) to the system $(\bullet)$ on the hermitian bundle $V \to S^2$: let G_λ be the corresponding loop (2.3). The factorisation (2.5) of G_λ into product of unitons, obtained in [V3] (using other words) is quite uncostructive, because it is based on the Birkhoff-Grothendieck decomposition theorem for holomorphic vector bundles over the 2-sphere. Here we show, among other things, that given a solution (A,Φ) of the system $(\bullet)$, the repeated addition of the kernel bundle (alternatively, the Image bundle) of Φ_z as uniton, always makes us reach the 0-energy solution, with $\Phi=0$, after a finite number of steps.

Lemma 3.1 Let (A,Φ) be a solution of $(\bullet)$ on $V \longrightarrow M^2$, and let $(\tilde{A}, \tilde{\Phi})$ be a new solution of $(\bullet)$, obtained by addition of a uniton $p \subseteq V$. Then we have:

(i) $\mathrm{Tr}\,(\Phi_{z}\tilde{\ })^k = \mathrm{Tr}\,(\Phi_z)^k \ \ \forall k$

(ii) $\Phi_{z}\tilde{\ } p^\perp = p^\perp \Phi_z, \quad p\,\Phi_{z}\tilde{\ } = \Phi_z p$

Proof.

By(13) in [V3], we have,

$$\Phi_{z}\tilde{\ } = -(p^\perp \partial_A p) + (p\Phi_z p) + (p^\perp \Phi_z p^\perp)$$

Therefore:

$$\mathrm{Tr}\ (\Phi_{z}\tilde{\ })^k = \mathrm{Tr}\left\{(p^\perp \partial_A p)^k + (p\Phi_z p)^k + (p^\perp \Phi_z p^\perp)^k\right\} = \mathrm{Tr}\left\{ (p\Phi_z p)^k + (p^\perp \Phi_z p^\perp)^k\right\}$$

$$= \mathrm{Tr}\left\{ p\,(\Phi_z)^k + p^\perp (\Phi_z)^k \right\} = \mathrm{Tr}\,(\Phi_z)^k$$

Moreover, using the uniton equations, we get:

$$-\Phi_{\tilde z}\, p^\perp = p^\perp \Phi_z p^\perp = p^\perp \Phi_z, \qquad p\Phi_{\tilde z} = p\Phi_z p = \Phi_z p$$

□

Remarks.

We know (cf. [U], [H]) that , for each k, $\mathrm{Tr}\,(\Phi_z)^k$ is a holomorphic k-differential on the compact Riemann surface M^2 (this means that, modulo passing to some branched covering of M^2 (cf. [H]), the eigenvalues of Φ_z are holomorphic 1-forms). Lemma 3.1 (i) means that adding unitons do not change the spectrum of Φ_z.

In particular, on S^2 there are no non-zero holomorphic k-differentials, therefore Φ_z is pointwise nilpotent. More generally, if we have a solution (A,Φ) of (*) which is factorisable into unitons, then Φ_z must be nilpotent, by lemma 3.1. This suggest the idea that "not many"(in a heuristic sense) solutions of (*) on a generic Riemann Surface are product of unitons (cf. also [V3]).

If we are anyway on S^2, then the image and kernel bundles $\mathrm{Im}\,\Phi_z$, $\mathrm{Ker}\,\Phi_z$ are well defined (up to a set of isolated points, where they may be regularised, cf. [U]); lemma 3.1 (ii) implies then:

$$\underline{p} \subseteq \mathrm{Ker}\,\Phi_z \Longleftrightarrow \mathrm{Im}\,\Phi_{\tilde z} \subseteq \underline{p}^\perp \qquad (3.1)$$

$$\mathrm{Im}\Phi_z \subseteq \underline{p} \Longleftrightarrow \underline{p}^\perp \subseteq \mathrm{Ker}\,\Phi_{\tilde z} \qquad (3.2)$$

This is the base of the following:

Proposition 3.2. Let (A, Φ) be a solution of (*) on $V \to S^2$. Then if we keep adding the uniton $\underline{p} = \mathrm{Ker}\,\Phi_z$, we eventually get the 0-energy solution, with $\Phi = 0$.

Remark. As it will result obvious from the proof, the same statement holds for the Image bundle $\mathrm{Im}\Phi_z$.

Proposition 3.2 has also been obtained independently by J.C.Wood.

Proof.

If K is the canonical line bundle of (1,0) forms over S^2, then Φ_z induces holomorphic section of:

$K \otimes \mathrm{Hom}(V/\mathrm{Ker}\Phi_z, \mathrm{Im}\Phi_z)$

Passing to the top exterior power, as in [W], we get:

$0 \leq c_1(\mathrm{Im}\Phi_z) + c_1(K) - c_1(V/\mathrm{Ker}\Phi_z) = c_1(\mathrm{Im}\Phi_z) + c_1(\mathrm{Ker}\Phi_z) - c_1(V) - 2$

But , modulo some trivial computations, this is equivalent to:

$\sigma(\mathrm{Im}\Phi_z) + \sigma(\mathrm{Ker}\,\Phi_z) \leq -2$ **(3.3)**

(where, for $\underline{p} \subseteq V$ complex subbundle we define $\sigma(\underline{p}) = \mathrm{rk}\,\underline{p}\,(\,\mu(V) - \mu(\underline{p})\,)$, as in [V3]).

Moreover, by (3.1), if we add the uniton $\underline{p} = \mathrm{Ker}\,\Phi_z$ to (A,Φ), and we get a solution $(\tilde{A}, \tilde{\Phi})$, then there are two possible cases:

(i) $\mathrm{rk}\,\Phi_z = \mathrm{rk}\,\tilde{\Phi}_z$, and $\underline{p}^{\perp} = \mathrm{Im}\,\tilde{\Phi}_z$;

(ii) $\mathrm{rk}\,\tilde{\Phi}_z < \mathrm{rk}\,\Phi_z$, $\underline{p}^{\perp} \supsetneq \mathrm{Im}\,\tilde{\Phi}_z$

Suppose we are in the case(i): then we have also $\sigma(\mathrm{Ker}\,\Phi_z) = \sigma(\mathrm{Im}\,\tilde{\Phi}_z)$, because the two bundles have the same 1st Chern class and rank . Therefore:

$\sigma(\mathrm{Ker}\,\tilde{\Phi}_z) - \sigma(\mathrm{Ker}\,\Phi_z) = \sigma(\mathrm{Ker}\,\tilde{\Phi}_z) + \sigma(\mathrm{Im}\,\tilde{\Phi}_z) \leq -2$ i.e.

$\sigma(\mathrm{Ker}\,\tilde{\Phi}_z) \leq -2 + \sigma(\mathrm{Ker}\,\Phi_z)$ **(3.4)**

Therefore, because of the energy formula in [V3]:

$$E(\tilde{\Phi}) - E(\Phi) = 8\pi\,\sigma(\mathrm{Ker}\,\Phi_z)$$

the repeated choice of $\mathrm{Ker}\,\Phi_z$ as uniton either decreases the rank of Φ_z, or it eventually makes $\sigma(\mathrm{Ker}\,\Phi_z)$ to be negative, and therefore the energy of the solution to decrease. In one way or another, we must reach a solution with $\Phi = 0$, in a finite number of steps.

□

4. Some remarks on the "extended solution".

Let now (A, Φ) be a solution of $(*)$ on $V \longrightarrow S^2$, and let $A_\lambda = A + \lambda\,\Phi_z + \lambda^{-1}\,\Phi_{\bar z}$ ($\lambda \in \mathbb{C}^*$) be the corresponding Uhlenbeck loop of connections with constant central curvature. Let G_λ be the loop (2.3) $G_\lambda : S^1 \longrightarrow \mathcal{G}^{\mathbb{C}}$, $G_\lambda * (A_0) = A_\lambda$. By proposition (2.3) we have a finite Fourier expansion:

$$G_\lambda = \sum_{0\le\alpha\le H} \lambda^\alpha\, T_\alpha \qquad \lambda \in \mathbb{C}^* \qquad \text{with } T_\alpha \in \mathfrak{g}. \tag{2.4}$$

From the unitarity of G_λ, for $\lambda \in S^1$, we get:

$$\sum_{0\le\alpha\le H}' T_\alpha^*\ T_{\alpha+k} = \sum_{0\le\alpha\le H} T_{\alpha+k}\, T_\alpha^* = I\ \delta_{0,k} \tag{4.1}$$

In particular, we have :

$$T_0^*\, T_H = T_H\ T_0^* = 0 \tag{4.2}$$

$$T_H^*\, T_0 = T_0\ T_H^* = 0 \tag{4.3}$$

Moreover, we get from (2.4), and (2.3):

$$\bar\partial_A T_\alpha + \Phi_{\bar z}\, T_\alpha = T_{\alpha+1}\, \Phi_{\bar z} \tag{4.4}$$

$$\bar\partial_A T_\alpha^* - T_\alpha \Phi_z = -\Phi_{\bar z} T_{\alpha-1} \tag{4.5}$$

From the above formulas we can get lot of interesting information.

For example, T_H is a holomorphic map between holomorphic bundles:

$$T_H : (V, \bar\partial_A) \longrightarrow (V, \bar\partial_{A_0}) \tag{4.6}$$

Analogously, T_0^* is holomorphic as a map:

$$T_0^* : (V, \bar\partial_{A_0}) \longrightarrow (V, \bar\partial_A) \tag{4.7}$$

Therefore: $\operatorname{Ker} T_0^* \supseteq \operatorname{Im} T_H$ are $\bar\partial_{A_0}$-holomorphic subbundles of $V \longrightarrow S^2$.

Moreover

$$\mathrm{Ker}\,\Phi_z \supseteq \mathrm{Im}\,T_0^* \quad \subseteq \quad \mathrm{Ker}\,T_H \supseteq \mathrm{Im}\,\Phi_z$$

are $\bar{\partial}_A$- holomorphic, Φ_z- stable subbundles of $V \longrightarrow S^2$, and hence possible unitons.

If we call $\underline{p}$ (respectively $\underline{p}^+$) the subbundle of $V \longrightarrow S^2$ generated by the sum of those line bundles L's in the Birkhoff- Grothendieck decomposition of of $(V, \bar{\partial}_A)$ with $\sigma(L) \geq 0$ (resp. $\sigma(L)>0$) then we must have, because of the above equations,

$$\mathrm{Ker}\,T_H \supseteq \underline{p}^+, \text{ and } \underline{p} \supseteq \mathrm{Im}\,T_0^*$$

In particular, $\sigma(\mathrm{Ker}\,T_H) < 0$, and adding the uniton Ker T_H is energy decreasing.

We remark that, modulo change of notation, Ker T_H is the uniton used by Uhlenbeck in her factorisation theorem. Therefore <u>Uhlenbeck' s factorization in [U] is energy decreasing</u>.

Other holomorphic invariants may be obtained from considering the expansion (2.4). As an example, the flag of subbundles of $V \longrightarrow S^2$:

$$\begin{aligned} &\mathrm{Im}\,T_H \\ &\mathrm{Im}\,T_H + \mathrm{Im}\,T_{H-1} \\ &\ldots\ldots \\ &\mathrm{Im}\,T_H + \mathrm{Im}\,T_{H-1} + \quad + \mathrm{Im}\,T_0 \end{aligned} \tag{4.8}$$

is $\bar{\partial}_{A_0}$-holomorphic.

For the sake of simplicity, let us suppose we are in the untwisted case: $V = S^2 \times \mathbb{C}^N$, $A_0 = 0$. Then (4.8) associates to every extended solution G_λ, coming from an harmonic map $f: S^2 \longrightarrow U(N)$, a holomorphic map from S^2 into a flag manifold.

Another holomorphic map into a flag manifold can be easily constructed using the kernel bundles of the T_α^*. It's not clear how much this construction is reversible, but it' s easy to prove that at least both the two maps are necessary. Moreover, this association of holomorphic maps into flag manifolds to harmonic maps $S^2 \longrightarrow U(N)$ looks very similar to recent work by J.C.Wood (cf. [W]), and it might be worth to compare the two approaches.

5. Some remarks about moduli spaces, and a spectral construction.

We have seen that every solution of the system (*) on $V \longrightarrow S^2$ may be described in terms of a discrete family of holomorphic objects (the unitons).

More generally, on curves of greater genus, we expect the moduli space to be something continuous, hopefully a differentiable manifold, once we factor out all uniton transformations.

We introduce an equivalence relation on the space of solutions (A,Φ) of the system of equations (*) on a hermitian vector bundle $V \longrightarrow M^2$, expressed as loop of connections with constant central curvature .

We say the two solutions A_λ, B_λ are equivalent if there exist a loop of gauge transformation G_λ, which are unitary for $|\lambda| = 1$, that transform A_λ into B_λ:

$\exists\, G_\lambda : \mathbb{C}^* \longrightarrow \mathcal{G}^c$, such that we have: $\qquad G_{\lambda *}(A_\lambda) = B_\lambda$

with G_λ unitary for $|\lambda| = 1$. **(5.1)**

We remark that (2.3)is a special case of (5.1) , when we take as B_λ a constant loop (i.e. a solution of (*) with energy 0).

The following two conjectures are, at this point, quite natural.

Conjecture1. Every such G_λ is product of unitons, as in (2.5).

Conjecture 2. Let us call X the space of solutions of the system (*); and let $\approx$

be the equivalence relation (5.1) on X: then the space $Y = X/_{\approx}$ is a (finite dimensional) smooth manifold.

Conjecture 1 is not too far away from being proved, using a generalisation of Uhlenbeck's argument in [U] (cf. our summary in § 2). It is possible indeed to prove that there do exist an elliptic differential operator L, such that the loop G_λ in (5.1) lies in the kernel of L, for each $\lambda \in \mathbb{C}^*$. It shouldn't be too difficult then to deduce, as in [U], the finiteness of the Fourier expansion in $\lambda = e^{it}$ of G_λ. The factorisation into unitons would be then an immediate corollary.

(We remark that, unfortunately, a kind of argument of algebro-geometric type, using energy reduction, like the one in [V1], [V3], does not seem to work: it is possible anyway to prove, using the 1-form ω (2.7), that the degree of the loop G_λ in (5.1) is given by the difference of the energies of A_λ and B_λ: but this does not seem to help anyway. We hope nevertheless this is not the case, and we will try further in this direction, in the future).

As much as it concerns conjecture 2, it does not look to be easily provable (perhaps because, being a conjecture, it may be false). Anyway, we remark the following.

Given a solution of our system (*), expressed as a loop of connections A_t, with constant central curvature, then, infinitesimally, the tangent space to the space of solutions is given by:

$$T_{A_t}(X) = \left\{ T_t = Q + \cos t\, R + \operatorname{sen} t\, *R \mid d_{A_t} T_t = 0\ \forall t \right\} \qquad \textbf{(5.2)}$$

where Q, R, are 1-forms, with coefficients in the skew-symmetric endomorphisms of $V \longrightarrow M^2$. We define the subspace :

$$J = \left\{ T_t \in T_{A_t}(X) \mid T_t = d_{A_t} u_t \quad u_t \text{ loop in the gauge lie algebra } \mathfrak{g} \right\} \quad \textbf{(5.3)}$$

Then we have :

$$T_{A_t}(Y) = T_{A_t}(X/_{\approx}) = T_{A_t}(X)/J \qquad \textbf{(5.4)}$$

Therefore the tangent space to our expected "moduli space" of solutions is very similar to a first cohomology group, with coefficients in the loop algebra $\Omega\,\mathfrak{u}(N)$.

If $M^2 = S^2$, then $\mathbb{J} = T_{A_t}(X)$: and, more in generally, most of the things described by Uhlenbeck in [U] on the space of Jacobi fields to a given harmonic map f: $S^2 \longrightarrow U(N)$ (most notably, the action of a Kac-Moody algebra) look to be generalisable to the space $\mathbb{J}$: this may be seen, for example, considering power series expansion in t.

Continuing with the "fried air" style of this §, now that we have a candidate Y for a moduli space, we will construct now an injective map:

$$Y \longrightarrow \text{Spectral (holomorphic) data} \tag{5.5}$$

Given a solution (A,Φ) of the system of equations (*) on a hermitian vector bundle $V \longrightarrow M^2$, let B_λ, u_λ be , for $\lambda \in \mathbb{C}$, solutions of the system:

$$\begin{aligned} &\Phi_z = \partial_{A_\lambda} u_\lambda + \bar{\lambda}\,[\Phi_z, u_\lambda] + B_\lambda \\ &\bar{\partial}_A B_\lambda + \lambda\,[\Phi_{\bar{z}}, B_\lambda] = 0 \end{aligned} \tag{5.6}$$

We remark that (5.6) is an Hodge-style decomposition of Φ_z into the sum of two orthogonal pieces, which are uniquely detemined. In particular, the choice of B_λ is unique; from this one can deduce that the limit of B_λ, for $\lambda \longrightarrow 0$, is Φ_z itself.

Moreover, we can interpretate $\left((-A_\lambda)^*, B_\lambda \right)$ as a map from the λ-plane $\mathbb{C}$ into the cotangent space of the space of holomorphic vector bundles over M^2. Consequently, we can apply Hitchin's construction in [H2], and we may consider, for each $\lambda \in \mathbb{C}$, the N-fold branched covering $\pi: M^2_\lambda \longrightarrow M^2$

$$M^2_\lambda = \left\{ (z,\mu) \in T^*_{(1,0)}(M^2) \mid \det(B_\lambda - \mu I) = 0 \right\} \tag{5.7}$$

and the $\bar{\partial}_{(-A_\lambda)^*}$ holomorphic vector bundle (which is "generically" a line bundle):

$L_\lambda \longrightarrow M^2_\lambda$ given by:

$$L_\lambda = \left\{ \mathrm{Ker}\,(B_\lambda - \mu I) \right\} \subset \pi^* (V \longrightarrow M^2,) \qquad (5.8)$$

considered as an element of the Jacobian of $M^2{}_\lambda$.

The interesting thing about this construction is that it is invariant under the gauge group action on the space of solutions of (*), and under the equivalence relation $\approx$ (5.1). Moreover, it is possible to prove the following: if two solutions A_λ, and B_λ produce the same spectral data (5.7) and (5.8), then they are $\approx$-equivalent (and therefore the map (5.5) is injective). Indeed, applying the results in [H2], we get that A_λ, and B_λ must be equivalent under a loop of complex gauge transformations. If we restrict ourselves to the case $|\lambda| = 1$, then the unitarity of the connections A_λ, B_λ, and the constant central curvature condition imply (as in the proof of the uniqueness of the Einstein connection for a stable bundle, in [D]) that the connecting gauge transformation may be chosen real.

The space of "spectral data"(loops of branched coverings + line bundles) is infinite dimensional, while the space Y of solutions of our system is finite dimensional, by general properties of elliptic systems. Therefore the problem is to characterise the image of the map (5.1):

$Y \longrightarrow$ Spectral (holomorphic) data

so that an inversion of it, possibly in a explicit way, becomes possible.

For a succeeded operation of this type, we refer the reader to recent work by Hitchin, on the algebro-geometric construction of all harmonic maps $f: T^2 \longrightarrow SU(2) = S^3$.

6 Holomorphic maps from a Riemann surface into the loop group $\Omega U(N)$.

We want to apply some of the previous constructions to study holomorphic map from compact Riemann surfaces into the loop group $\Omega U(N)$. (This generalizes the case when the holomorphic map is given by Uhlenbeck's extended solution G_λ).

It turns out that there do exist "uniton transformations"; but that only a class of holomorphic maps $M^2 \longrightarrow \Omega U(N)$ are product of unitons (even if we are in the best case: $M^2 = S^2$).

Let $f: M^2 \longrightarrow \Omega U(N)$ be an holomorphic map. By § 1 this means that:

$$f^{-1}\partial f = \sum_{\alpha>0} (1-\lambda^\alpha) A_\alpha \tag{6.1}$$

has only positive terms in its Fourier expansion (after complexification of the loop variable).

Moreover we have:

$$2\pi \deg f = \sum_{\alpha>0} \alpha \, |A_\alpha|^2 \tag{6.2}$$

We consider a new map of the form

$$\tilde{f} = f\,(p^\perp + \lambda\, p\,) \tag{6.3}$$

where p is (as usual) the hermitian projection operator onto a subbundle $\underline{p}$ of $M^2 \times \mathbb{C}^N$.

Proposition 6.1.

(i) The map $\tilde{f} = f\,(p^\perp + \lambda\, p\,)$ is holomorphic if and only if $\underline{p}$ is holomorphic with respect to the $\bar\partial$-operator $\bar\partial + (-\sum_{\alpha>0} A_\alpha)^*$

(ii) $\deg \tilde{f} - \deg f = -c_1(\underline{p})$

Proof.

The proof is essentially as in [V3] . Let $\tilde{A} = \tilde{f}^{-1} d\tilde{f}$; with $\tilde{A}_z = \tilde{f}^{-1}\partial\tilde{f} = \sum_{\alpha>0} (1-\lambda^\alpha)\tilde{A}_\alpha$; and $A = f^{-1}df$, with $A_z = f^{-1}\partial f = \sum_{\alpha>0} (1-\lambda^\alpha) A_\alpha$;

let $B = B_z + B_{\bar z} = \sum_{\alpha>0} A_\alpha - (\sum_{\alpha>0} A_\alpha)^*$

Then $\tilde{T}$ is holomorphic if and only if we have $\tilde{A}_\alpha = 0$ for $\alpha<0$. But we have:

$$\tilde{A} = \sum_{\alpha>0}\Big((1-\lambda^\alpha)(pA_\alpha p + p^\perp A_\alpha p^\perp) + (\lambda^\alpha - \lambda^{\alpha+1})(p^\perp A_\alpha p) + $$
$$+ (\lambda^{-1} - \lambda^{\alpha-1})\, pA_\alpha p^\perp \Big) + (\lambda-1)\, p^\perp \partial p + (1-\lambda^{-1})\, p\partial p =$$
$$= \sum_{\alpha>0}\Big((1-\lambda^\alpha)(pA_\alpha p + p^\perp A_\alpha p^\perp) + (1-\lambda^{\alpha+1})\, p^\perp A_\alpha p + (1-\lambda^{\alpha-1})\, pA_\alpha p^\perp\Big) +$$
$$+ (\lambda-1)\Big(\sum_{\alpha>0} p^\perp A_\alpha p + p^\perp \partial p\Big) + (\lambda^{-1}-1)\Big(\sum_{\alpha>0} pA_\alpha p^\perp + p\partial p^\perp\Big)$$

Therefore, we have that $\tilde{T}$ is holomorphic if and only if

$$p^\perp \bar\partial_B p = 0.$$

(ii) We consider B as a connection on the trivial bundle $M^2 \times \mathbb{C}^N$. Its curvature is:

$$F(B) = \bar\partial B_z + \partial B_{\bar z} + \tfrac{1}{2}[B, B]$$

Substituting (6.1) in the condition that the connection $A = f^{-1}df$ must have curvature 0, we get:

$$\partial B_{\bar z} + \tfrac{1}{2}[B,B] = -\sum_{\gamma<\beta} [A_\beta^*, A_\gamma]$$
$$\bar\partial B_z + \tfrac{1}{2}[B,B] = -\sum_{\gamma>\beta} [A_\beta^*, A_\gamma]$$

Adding the two, we get:

$$F(B) = \sum [A_\alpha^*, A_\alpha] \qquad (6.4)$$

If p is the hermitian projection onto a subbundle $\underline{p} \subseteq M^2 \times \mathbb{C}^N$ then we get, if we argue like in [V3]:

$$\int_{M^2} \mathrm{Tr}\Big(F(B)p + d_B p \wedge d_B p\, p\Big) = -2i\pi\, c_1(\underline{p}) \qquad (6.5)$$

But we have, as in [V3]:

$$\int_{M^2} \mathrm{Tr}\Big(d_B p \wedge d_B p\Big)p = i\,|p^\perp \partial_B p|^2 - i\,|p^\perp \bar\partial_B p|^2$$

and, because of (6.4):

$$\int_{M^2} \mathrm{Tr}\Big(F(B)p\Big) = \int_{M^2} \mathrm{Tr}\Big(\sum [A_\alpha^*, A_\alpha]\, p\Big) = i\sum\Big(|p^\perp A_\alpha p|^2 - |pA_\alpha p^\perp|^2\Big)$$

Therefore we have:

$$i\,|\partial_B p|^2 - i\,|\bar\partial_B p|^2 + i \sum \left(|p^\perp A_\alpha p|^2 - |p A_\alpha p^\perp|^2 \right) = -2i\pi\, c_1(\underline{p}) \tag{6.6}$$

Let's return to the uniton transformation in part (i). We have, if $\tilde{f} = f\,(p^\perp + \lambda\, p)$:

$$2\pi \deg(\tilde{f}) = \sum_{\alpha>0} \alpha\, |\tilde{A}_\alpha|^2 = \sum \alpha \left(|pA_\alpha p|^2 + |p^\perp A_\alpha p^\perp|^2 \right) +$$
$$+ \sum |\alpha - 1|\ |pA_\alpha p^\perp|^2 + \sum |\alpha+1| (|p^\perp A_\alpha p|^2 + |p^\perp \partial_B p|^2)$$

Using then (6.6) and the holomorphicity condition for $\mathbf{p}$, we get:

$$2\pi \deg(\tilde{f}) = \sum_{\alpha>0} \alpha\, |A_\alpha|^2 +$$
$$+ \sum \alpha \left(|pA_\alpha p|^2 + |p^\perp A_\alpha p^\perp|^2 + |p^\perp A_\alpha p|^2 + |pA_\alpha p^\perp|^2 - |A_\alpha|^2 \right) +$$
$$+ \sum \left(- |pA_\alpha p^\perp|^2 + |p^\perp A_\alpha p|^2 + |p^\perp \partial_B p|^2 - |p^\perp \bar\partial_B p|^2 \right) =$$
$$= 2\pi \deg(f) - 2i\pi\, c_1(\underline{p})$$

(because the second $\sum$ is 0).

□

<u>Remark</u>. Similar statements hold for "generalized equations", in the spirit of [V3]; loops of connections A_λ on a hermitian vector bundle $V \longrightarrow M^2$, of the form:

$A_z = \sum_{\alpha>0} (1-\lambda^\alpha) A_\alpha$, satisfying the equation:

$*F(A_\lambda) = -2i\pi\, \mu(V)$

We leave to the reader to find the analogue of Prop. 6.1, and of the following.

Suppose we want to use prop. 6.1 in order to describe holomorphic maps $f: M^2 \longrightarrow \Omega U(N)$ as products of unitons, using energy reduction, as in [V1], [V3]. We remark that, if $f: M^2 \longrightarrow \Omega U(N)$ is an holomorphic map, then its energy is 2 times its degree (cf. § 1); therefore, if f has degree 0, then it must be constant. By proposition 6.1, given $f: M^2 \longrightarrow \Omega U(N)$ holomorphic, then there exixts an $\tilde{f} = f\,(p^\perp + \lambda\, p)$ which is holomorphic, with $0 \le \deg(\tilde{f}) < \deg f$ if and only if the bundle $M^2 \times \mathbb{C}^N$, with the complex structure induced by $\bar\partial + (-\sum_{\alpha>0} A_\alpha)^*$, is not semistable.

This gives no hope of having a general factorization theorem, because "most " of the

$\bar{\partial}$-operators give stable holomorphic structures.

Anyway, if we are in the case $M^2 = S^2$ then the only semistable vector bundles of 1st Chern class 0 are the trivial ones; therefore, we may prove the following.

Corollary 6.2 Let $f: S^2 \longrightarrow \Omega U(N)$ be an holomorphic map. Then the following statements are equivalent.

(i) f is product of unitons:

$$f = f_0 (p_1{}^{\perp} + \lambda\, p_1) \qquad (p_k{}^{\perp} + \lambda p_k)\ ,\ \text{wit } f_0 \in \Omega U(N).$$

(ii) The expansion $f^{-1}\partial f = \sum_{1 \le \alpha \le H} (1-\lambda^{\alpha}) A_{\alpha}$

has a finite number of terms.

Proof.

(i) $\Rightarrow$ (ii) is trivial.

Conversely, because of the remarks after proposition 6.1, given $f: S^2 \longrightarrow \Omega U(N)$ holomophic, we can add unitons to f, producing new holomorphic maps of strictly smaller degree, as long as the $\bar{\partial}$-operator $\bar{\partial}_{-(\sum_{\alpha>0} A_{\alpha})^{*}} = \bar{\partial}_B$ is not trivial. Suppose we finally reach a map f, with $\bar{\partial}_B$ trivial. Then, from the 0-curvature condition $F(f^{-1} df) = 0$, we get, if A_H is the top non zero-term in the Fourier expansion of $f^{-1}\partial f$:

$$\bar{\partial}_B (A_H) = 0$$

Because of the triviality of the holomorphic structure on $S^2 \times \mathbb{C}^N$ induced from $\bar{\partial}_B$, and of the fact that there do not exist holomorphic 1-forms on the Riemann sphere, this implies $A_H = 0$, which is a contradiction. Therefore, if $\bar{\partial}_B$ is trivial, and the expansion of $f^{-1}\partial f$ is finite, then f is a constant map, and B = 0.

□

Remarks.

(Based) holomorphic maps $M^2 \longrightarrow \Omega U(N)$ correspond to holomorphic bundles over $M^2 \times S^2$, (with some triviality properties: cf. [A]). It would be interesting then to have an interpretation of the flag transform in Proposition 6.1, in terms of these holomorphic bundles.

In particular, based holomorphic maps $S^2 \longrightarrow \Omega G$ are in 1-1 correspondance with the space of G-instantons (G classical Lie group) over $\mathbb{R}^4$, modulo based gauge transformations (cf. [A]). Therefore the question of how many holomorphic maps $S^2 \longrightarrow \Omega U(N)$ are factorizable into a product of unitons, which has only partially been answered in Corollary 6.2, is particularly interesting (cf. also [Hu] for a study of holomorphic maps $S^2 \longrightarrow SU(2)$).

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